26 September 95

Last time, we saw how to solve for the potential in the channel $0 \le x \le a$ given the boundary conditions:

$\Phi = 0$ along $x = 0$

$\Phi = 0$ along $x = a$

$\Phi \to 0$ as $y \to \infty$

RESERVE

and

$\Phi(x,0) = \Phi_0(x)$ (some arbitrary function)

The solution was:

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

where $$A_n = \frac{2}{a}\int_0^a dx\, \Phi_0(x) \sin\left(\frac{n\pi x}{a}\right)$$

Now let's look at a special case: $\Phi_0(x) = V_0$ that is, the potential is a constant at $y=0$, $0<x<a$. It turns out that the Fourier series can be summed into a closed expression in simple functions.

$$A_n = \frac{2}{a}\int_0^a dx\, V_0 \sin\left(\frac{n\pi x}{a}\right) = \frac{2V_0}{n\pi}\left(1 - \cos n\pi\right) = \begin{cases} 0, & n\text{-even} \\ \frac{4V_0}{n\pi}, & n\text{-odd} \end{cases}$$

$$\Phi(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_0}{\pi n} \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

$$= \frac{4V_0}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\left[\frac{(2k-1)\pi x}{a}\right] e^{-\frac{(2k-1)\pi y}{a}}$$

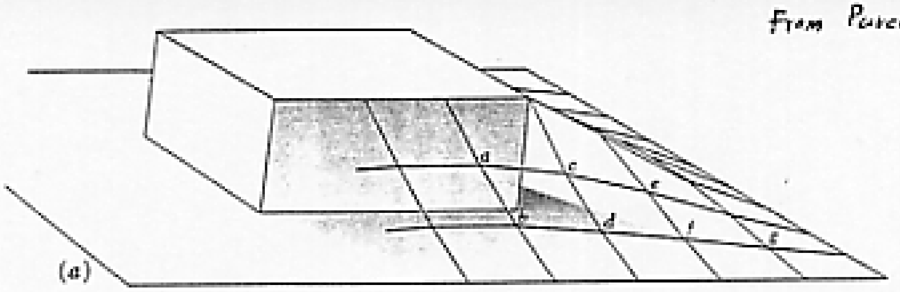
$$= \frac{2V_0}{\pi} \tan^{-1}\left[\frac{\sin\left(\frac{\pi x}{a}\right)}{\sinh\left(\frac{\pi y}{a}\right)}\right]$$

RESERVE

The Fourier Expansion converges uniformly (point-by-point) to this function everywhere except the two points $(0,0)$ and $(a,0)$ where the boundary conditions are discontinuous.

End Lecture #9

From Purcell



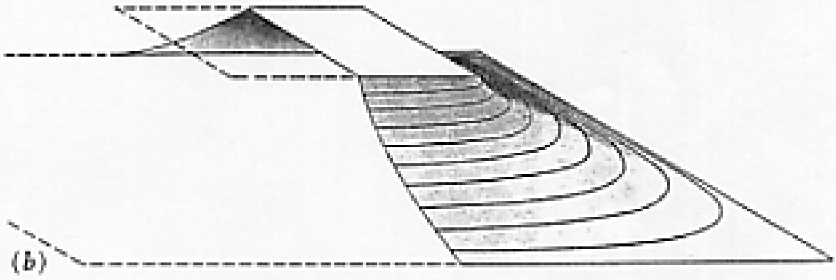
Prob. 3.31

3.31 The way the potential varies in the space between two charged cylinders has its exact counterpart in a different physical phenomenon, the shape assumed by an elastic membrane. Imagine the lattice of points illustrated in Prob. 3.29, including the boundary points, to be connected by rubber bands all stretched to the same tension. Now imagine the inner boundary elevated, as in the illustration for this problem, to a height representing the potential difference φ_0. Assuming all slopes are small enough so that angles $\approx$ tangents, etc., the equilibrium height of each junction point, or node, will be just the average of the heights of its four neighbors. Why?

Questions for discussion: The physical significance of the relaxation method

RESERVE

If we have a continuous elastic sheet instead of a net, the height of the surface obeys Laplace's equation and raising the inner frame causes the surface to assume the shape illustrated in part (*b*). This is exactly the solution $\varphi(x,y)$ for the electrostatic potential between two square cylinders, and it is also the shape of a soap film between two square frames, sketched in part (*c*). Visualizing the soap film or network of rubber bands sometimes helps one to anticipate the nature of a solution to a boundary-value problem in some other part of physics. That is how we guessed the starting values suggested for the relaxation calculation. Comparison of parts (*b*) and (*a*) of the figure, shows why we cannot expect a lattice solution to give us all the details. The steep fall of the potential in the immediate neighborhood of an inner corner—where in fact the electric field becomes infinite—could not be revealed.



With the variational principle in mind, we recognize that the elastic systems, sheets or bands, have adjusted themselves to minimize the elastic energy. In the case of the sheet or soap film, this means that the area of the surface is a minimum. The shape is simply that of the surface of least area joining the given boundaries.

What quantity is the counterpart, in the electrostatic system, of the total downward force on the inner frame, in the elastic system?

Think about the meaning of our relaxation calculation in mechanical terms. Imagine poles erected at each of the lattice points, with each node of the net clamped to its pole at some arbitrary height at the beginning. What process are we carrying out in the relaxation calculation?

from Griffiths

RESERVE

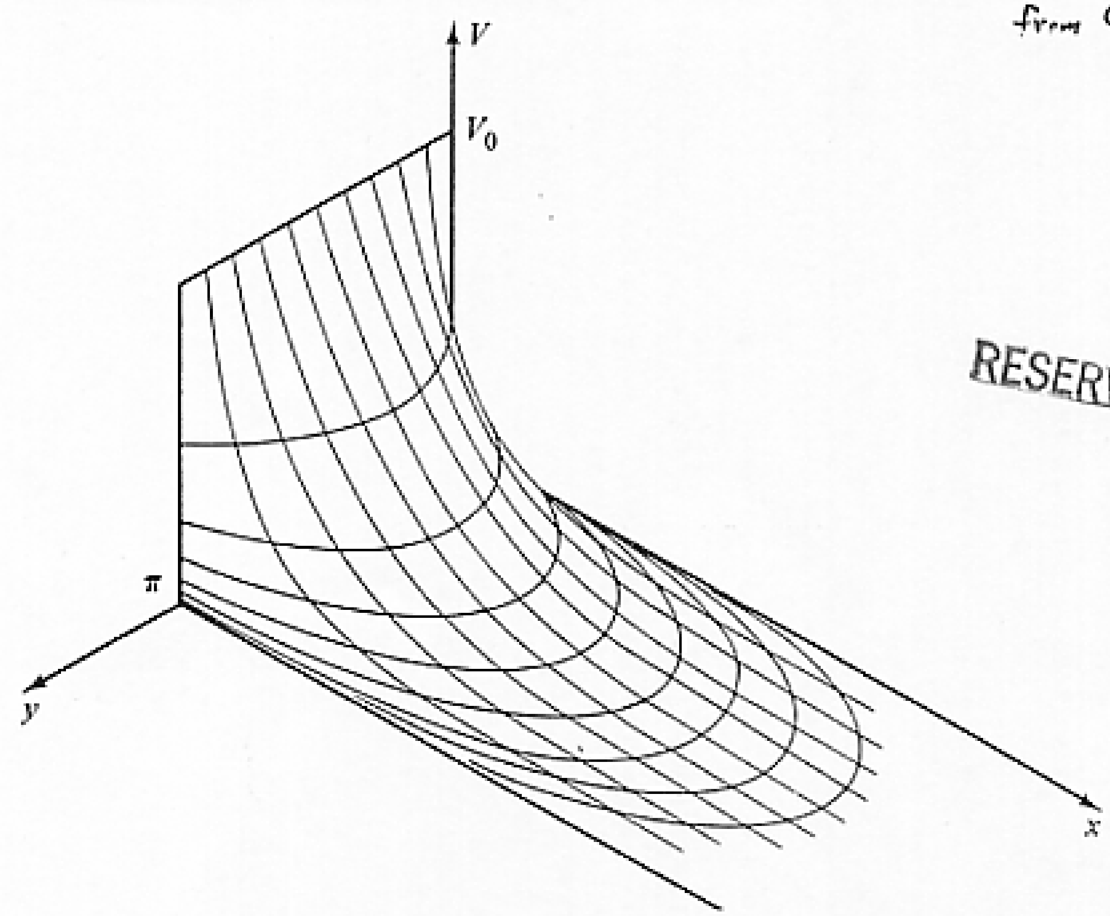


Figure 3.17 (From *Electromagnetic Fields and Waves*, Second Edition, Dale Lorrain and Paul R. Corson, W. H. Freeman and Company © 1970.)

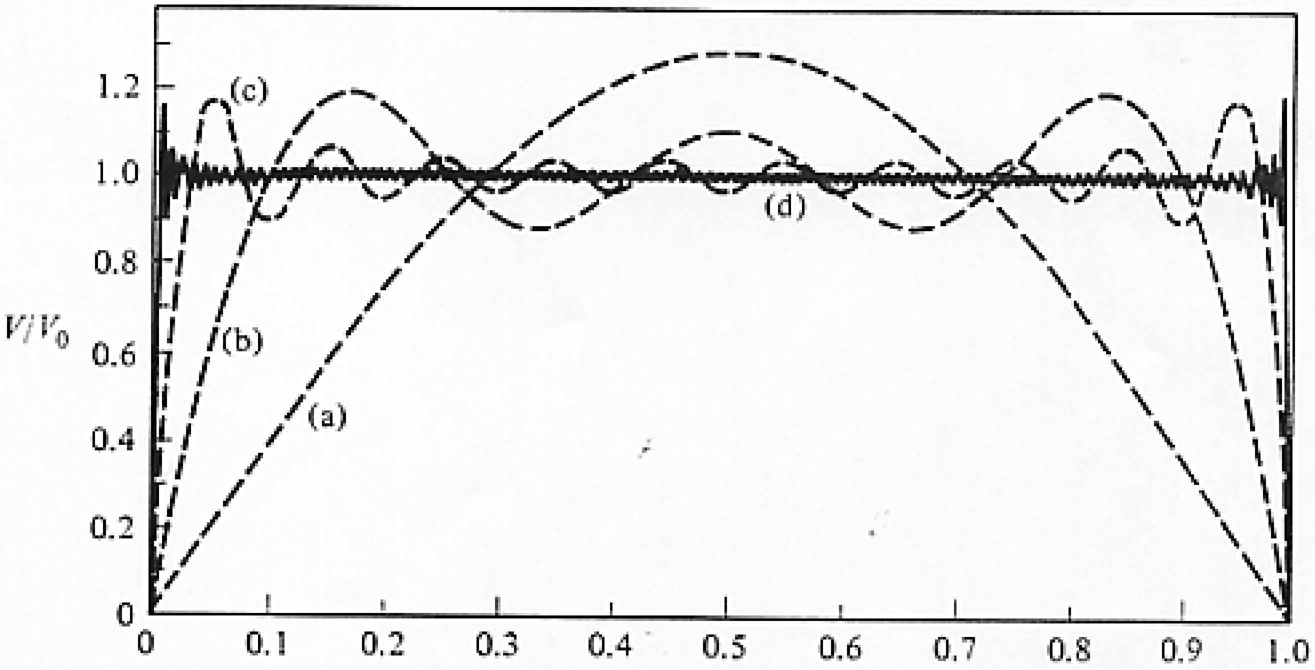


Figure 3.18 (From *Electromagnetic Fields and Waves*, Second Edition, Dale Lorrain and Paul R. Corson, W. H. Freeman and Company © 1970.)

In this form it is easy to check that Laplace's equation is obeyed and the four boundary conditions (3.30) are satisfied.

Boundary conditions (i) and (ii) require, as before, that $D = 0$ and k be a positive integer, leaving us with the separable solution

$$V(x, y) = C \cosh kx \sin ky. \tag{3.36}$$

Because $V(x, y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x, y) = \sum_{k=1}^{\infty} C_k \cosh kx \sin ky,$$

and use it to satisfy condition (iii):

$$V(1, y) = \sum_{k=1}^{\infty} C_k \cosh k \sin ky = V_0.$$

This is the same problem in Fourier analysis that we faced before: I quote the solution from equation (3.31):

$$C_n \cosh n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

RESERVE

Conclusion: The potential in this case is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{k=1,3,5\ldots} \frac{1}{k} \frac{\cosh kx}{\cosh k} \sin ky. \tag{3.37}$$

This function is shown in Fig. 3.20.

From Griffiths

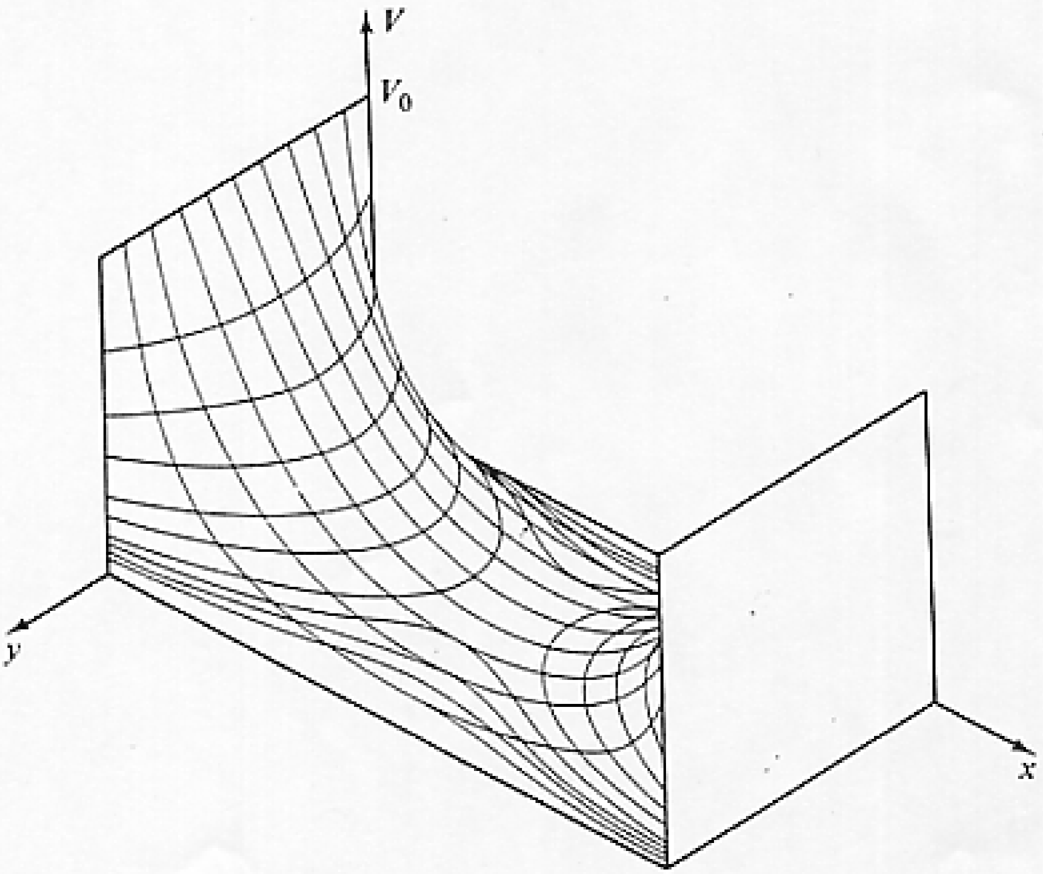


Figure 3.20 (From *Electromagnetic Fields and Waves*, Second Edition, Dale Lorrain and Paul R. Corson, W. H. Freeman and Company © 1970.)