5 October 95

Consider a parallelopiped of dimension $a \times b \times c$ with origin at one corner. The volume V is $(0 < x < a), (0 < y < b), (0 < z < c)$

The set of functions

$$u_{lmn}(\vec{r}) = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right)$$

are <u>orthonormal</u> in V:

$$\int_0^a dx \int_0^b dy \int_0^c dz \; u_{l'm'n'}(\vec{r})\, u_{lmn}(\vec{r}) = \delta_{ll'}\, \delta_{mm'}\, \delta_{nn'}$$

and <u>complete</u>, that is for any well-behaved function $f(\vec{r})$ that vanishes on the sides of the box:

$$f(\vec{r}) = \sum_{l}\sum_{m}\sum_{n=1}^{\infty} f_{lmn}\, u_{lmn}(\vec{r}) .$$

To solve for the expansion coefficients, we multiply both sides by $u_{l'm'n'}(\vec{r})$ and integrate over V.

RESERVE

$$\int_0^a dx \int_0^b dy \int_0^c dz \; u_{\ell' m' n'}(\vec{r})\, f(\vec{r}) = \sum_{\ell}\sum_{m}\sum_{n=1}^{\infty} f_{\ell m n} \int_0^a\!\int_0^b\!\int_0^c dV \; u_{\ell' m' n'}(\vec{r})\, u_{\ell m n}(\vec{r})$$

$$= \sum_{\ell}\sum_{m}\sum_{n=1}^{\infty} f_{\ell m n} \left(\delta_{\ell\ell'}\,\delta_{m m'}\,\delta_{n n'}\right) = f_{\ell' m' n'}$$

So the expansion coefficients are

$$f_{\ell m n} = \int_0^a dx' \int_0^b dy' \int_0^c dz' \; u_{\ell m n}(\vec{r}\,')\, f(\vec{r}\,')$$

where $\vec{r}\,' = (x', y', z')$ are dummy variables of integration.

If we substitute these coefficients back into the expression for $f(\vec{r})$ we will obtain a useful relation.

$$f(\vec{r}) = \int_0^a dx' \int_0^b dy' \int_0^c dz' \left[\sum_{\ell}\sum_{m}\sum_{n=1}^{\infty} u_{\ell m n}(\vec{r})\, u_{\ell m n}(\vec{r}\,') \right] f(\vec{r}\,')$$

the only way to satisfy this equation is if

$$\sum_{\ell}\sum_{m}\sum_{n=1}^{\infty} u_{\ell m n}(\vec{r})\, u_{\ell m n}(\vec{r}\,') = \delta^3(\vec{r} - \vec{r}\,')$$

This relation is called <u>closure</u>.

The closure relation and the orthonormality relation will look more similar if we treat the Fourier indices (ℓ, m, n) and the coordinates (x, y, z) on an equal footing:

orthonormality: $\sum_x \sum_y \sum_z \mathcal{U}_{\ell' m' n', xyz}\, \mathcal{U}_{\ell m n, xyz} = \delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}$

closure: $\sum_\ell \sum_m \sum_n \mathcal{U}_{\ell mn, xyz}\, \mathcal{U}_{\ell mn, x'y'z'} = \delta_{xx'}\delta_{yy'}\delta_{zz'}$

For later reference, let us consider the Laplacian of $\mathcal{U}_{\ell mn}(\vec{r})$:

$$\nabla^2 \mathcal{U}_{\ell mn}(\vec{r}) = -\pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\mathcal{U}_{\ell mn}(\vec{r})$$

Now we can try to find a Dirichlet Green function for the box geometry with origin at one corner. We demand:

1) $G_D(\vec{r}, \vec{r}\,') = G_D(\vec{r}\,', \vec{r})$ symmetry
2) $\nabla^2 G_D(\vec{r}, \vec{r}\,') = -4\pi\,\delta^3(\vec{r}, \vec{r}\,')$
3) $G_D(\vec{r}, \vec{r}\,') = 0$ for $\vec{r}$ or $\vec{r}\,'$ on S

We might guess

$$G_D(\vec{r},\vec{r}') = \sum_{l}\sum_{m=1}^{\infty}\sum_{n} G_{lmn}\, u_{lmn}(\vec{r})\, u_{lmn}(\vec{r}')$$

This is symmetric in $\vec{r}$ and $\vec{r}'$, and it vanishes on the boundary S because $u_{lmn}(\vec{r})$ was constructed out of sines which vanish on S.

We will use the coefficients G_{lmn} to satisfy
$\nabla^2 G_D(\vec{r},\vec{r}') = -4\pi\,\delta^3(\vec{r}-\vec{r}')$.

$$\nabla^2 G_D(\vec{r},\vec{r}') = \sum_{l}\sum_{m=1}^{\infty}\sum_{n} G_{lmn}\left[-\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)u_{lmn}(\vec{r})\right]u_{lmn}(\vec{r}')$$

We must have

$$G_{lmn} = \frac{4}{\pi}\,\frac{1}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

The Dirichlet Green function for the box with origin at one corner is:

$$G_D(\vec{r},\vec{r}') = \frac{4}{\pi}\sum_{l}\sum_{m=1}^{\infty}\sum_{n} \frac{u_{lmn}(\vec{r})\, u_{lmn}(\vec{r}')}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

or expanded out in full:

$$G_D(\vec{r},\vec{r}') = \frac{4}{\pi}\left(\frac{8}{abc}\right)\sum_{\ell}\sum_{m}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\ell\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{\ell\pi x'}{a}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\frac{\ell^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

Let's compare this to the Dirichlet Green function for the box obtained early by the method of images. Remember in the image problem, we put the origin at the center of the box, but a simple translation of coordinates will move it to a corner.

$$G_D^{image}(\vec{r},\vec{r}') = \sum_{\ell}\sum_{m}\sum_{n=-\infty}^{\infty} \frac{(-1)^{\ell+m+n}}{\sqrt{[x-(-1)^{\ell}x'+\ell a]^2+[y-(-1)^{m}y'+mb]^2+[z-(-1)^{n}z'+nc]^2}}$$

Recall that the Green function is the potential at $\vec{r}$ due to a unit point charge at $\vec{r}'$ (and vice versa!). But we have a uniqueness theorem for the electrostatic potential, so these two representations of the Green function must be equivalent.

$$G_D(\vec{r},\vec{r}') = G_D^{image}(\vec{r},\vec{r}')$$

For yet another representation, see Jackson, page 121. Here he sums over n and introduces an asymmetry in the z-direction.

b) Circular Cylindrical Coordinates + Separation of Variables

Recall that in general, Laplaces equation is

$$\nabla^2 \Phi = 0 = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial q_i}\left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial}{\partial q_i} \Phi \right)$$

where the h's can be functions of the generalized coordinates $\underline{q}$.

In Cartesian Coordinates: $h_1 = 1 = h_2 = h_3$

In Circular Cylindrical Coordinates: $h_\rho = 1$, $h_\varphi = \rho$, $h_z = 1$

So Laplace's equation is

$$\nabla^2 \Phi = 0 = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho} \Phi \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \Phi + \frac{\partial^2}{\partial z^2} \Phi$$

We look for solutions of the form

$$\Phi(\rho,\varphi,z) = R(\rho)\, F(\varphi)\, Z(z)$$

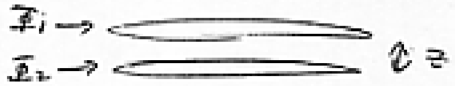
We will consider several special cases as we did with Cartesian boundary conditions:

i) The boundary conditions depend only on one coordinate

z) If the boundary conditions depend on z alone, then the potential can depend only on z.

$$\nabla^2 \Phi = 0 = \frac{\partial^2}{\partial z^2} Z(z) \Rightarrow Z(z) = C_1 z + C_2$$

We have seen this problem before. If ρ varies from 0 to infinity and φ varies over 0 to 2π, they will sweep out a plane at some fixed value of z. This is an infinite parallel-plate capacitor which we met while studying Cartesian boundary conditions that depend on one coordinate.

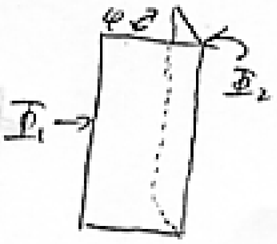


φ) If the boundary conditions depend on φ alone, the potential must be a function of φ alone.

$$\nabla^2 \Phi = 0 = \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} F(\varphi)$$

multiply by ρ^2: $\quad \frac{\partial^2}{\partial \varphi^2} F(\varphi) = 0 \Rightarrow F(\varphi) = C_1 \varphi + C_2$

If ρ and z vary over their ranges, we see that we are dealing with an infinite wedge.



Now the "new" solution:

ρ) If the boundary conditions depend on ρ alone, the potential must be a function of ρ only.

$$\nabla^2 \Phi = 0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho \frac{\partial}{\partial\rho} R(\rho)\right]$$

multiply by ρ

$$\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}R(\rho)\right] = 0 \Rightarrow \rho\frac{\partial}{\partial\rho}R(\rho) = \text{constant} = c_1$$

$$R(\rho) = c_1 \ln \rho + c_2$$

A surface of fixed ρ and varying z and φ is an infinite circular cylinder.

Suppose that we want to find the potential <u>inside</u> an infinite cylinder of radius a with Dirichlet boundary condition $\Phi(\rho=a) = \Phi_0$.

The solution inside is

$$\Phi(\rho) = R(\rho) = c_1 \ln\rho + c_2$$

since this interior region includes $\rho = 0$, we must set $c_1 = 0$ to avoid the infinite $\ln(0)$ divergence. So then

$\Phi(\rho) = c_2$ for <u>$\rho < a$</u>. The boundary condition determines c_2. $c_2 = \Phi_0$. The potential is constant inside the infinite cylinder!

If we want to find the potential <u>outside</u> an infinite cylinder of radius $\underline{a}$, the general solution is:

$$\Phi(\rho) = R(\rho) = c_1 \ln \rho + c_2 \qquad \underline{\rho > a}$$

The two constants c_1 and c_2 are determined from the boundary conditions at $\rho = a$ and $\rho = \infty$: $\cancel{\Phi(a) = \Phi_0}$, $\cancel{\Phi(\infty) = 0.}$

We can also find the potential <u>between</u> two infinite cylinders of radii $\underline{a}$ and $\underline{b}$.

$$\Phi(\rho) = c_1 \ln \rho + c_2$$

c_1 and c_2 are determined from the Dirichlet boundary conditions;

$$\Phi(a) = \Phi_a \quad , \quad \Phi(b) = \Phi_b$$

ii) The boundary conditions depend on ρ and φ,

We look for solutions of the form

$$\Phi(\rho,\varphi) = R(\rho)\, F(\varphi)$$

Laplace's Equation is

$$\nabla^2 \Phi(\rho,\varphi) = 0 = \frac{1}{\rho}(\rho R')'F + \frac{1}{\rho^2} R F''$$

These variables will separate if we divide by RF/ρ^2.

$$\frac{\rho(\rho R')'}{R} + \frac{F''}{F} = 0$$

This is only a function of ρ

This is only a function of φ

The only way this can be true is if they are both equal to constants.

There are three sub-cases:

1) $C = 0$

$F'' = 0 \quad \Rightarrow F(\varphi) = c_1\varphi + c_2$

$(\rho R')' = 0 \quad \Rightarrow R(\rho) = d_1 \ln\rho + d_2$

2) $C = \alpha^2 \quad (\alpha > 0,\ \text{real})$

$F'' + \alpha^2 F = 0 \quad \Rightarrow F(\varphi) = c_1' \sin(\alpha\varphi) + c_2' \cos(\alpha\varphi)$

$\rho(\rho R')' - \alpha^2 R = 0 \quad \Rightarrow R(\rho) = d_1' \rho^{\alpha} + d_2' \rho^{-\alpha}$

3) $C = -\alpha^2 \quad (\alpha > 0,\ \text{real})$

$F'' - \alpha^2 F = 0 \quad \Rightarrow F(\varphi) = c_1'' \sinh(\alpha\varphi) + c_2'' \cosh(\alpha\varphi)$

$\rho(\rho R')' + \alpha^2 R = 0 \quad \Rightarrow R(\rho) = d_1'' e^{i\alpha \ln\rho} + d_2'' e^{-i\alpha\ln\rho}$

$= d_1'' \rho^{i\alpha} + d_2'' \rho^{-i\alpha}$

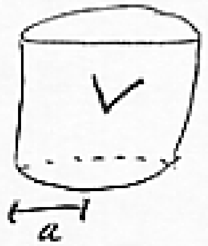
This differs from the Cartesian case in which $C = 0$ gave us only the null solution $\Phi(x,y,z) = 0$.

The general solution is therefore:

$$\Phi(\rho,\varphi) = [c_1 \varphi + c_2][d_1 \ln\rho + d_2]$$
$$+ [c_1' \sin(\alpha\varphi) + c_2' \cos(\alpha\varphi)][d_1' \rho^{\alpha} + d_2' \rho^{-\alpha}]$$
$$+ [c_1'' \sinh(\alpha\varphi) + c_2'' \cosh(\alpha\varphi)]\underbrace{[d_1'' \rho^{i\alpha} + d_2'' \rho^{-i\alpha}]}_{\text{or } [D_1'' \sin(\alpha \ln\rho) + D_2'' \cos(\alpha \ln\rho)]}$$

but we will see that for a specific problem, many of the coefficients must vanish.

1) Suppose the volume of interest is the <u>interior</u> of an infinitely long <u>full</u> ($\varphi = 0 \to 2\pi$) circular cylinder of radius a.



Since $\rho = 0$ is included in V we must have $d_1 = 0 = d_2'$ because: $\ln\rho$ and $\rho^{-\alpha}$ blow up at $\rho = 0$

Since F must be periodic of period 2π in the cylinder $F(\varphi) = F(\varphi + 2\pi)$ we must have

$c_1 = 0$ and $c_1'' = 0 = c_2''$

further, we must have $\alpha = n$, $n = 1, 2, 3 \ldots$

Inside the cylinder, the most general solution is:

$$\Phi(\rho,\varphi) = A_0 + \sum_{n=1}^{\infty} \rho^n \left[A_n \cos(n\varphi) + B_n \sin(n\varphi)\right]$$

The coefficients A_0, A_n, B_n $(n=1,2,\ldots)$ are determined from a Fourier analysis of the boundary conditions.

In this case, the boundary condition might be

$$\Phi(a,\varphi) = f(\varphi)$$ on the lateral surface of the cylinder, $\rho = a$.

— End Lecture # 11 —