21 November 95

First, a look back at the two common behaviors of substances in electrostatic + magnetostatic fields:

Place a <u>para</u>-electric substance in an electric field

Positive charge is attracted to the "lee-ward" side and negative charge to the "wind-ward" side

The polarization $\vec{P}$ points from the negative charge to the positive, like the electric dipole vector, since $\vec{P}$ is the electric dipole moment per unit volume.

Then using $\vec{D} = \vec{E} + 4\pi\vec{P}$ we see that the displacement field $\vec{D}$ is <u>enhanced</u> in a <u>para</u>-electric substance compared to vacuum.

Now place a <u>dia</u>-magnetic substance in a magnetic field - (a free-electron model will work here).

Using the right-hand rule, we see that the magnetic dipole moment of the current loop is directed opposite to the $\vec{B}$ field, so $\vec{M}$ the magnetization which is the magnetic dipole moment per unit volume also points in the direction opposite to $\vec{B}$.

Using $\vec{H} = \vec{B} - 4\pi\vec{M}$ we see that the intensity $\vec{H}$ is <u>enhanced</u> in a <u>dia</u>-magnet. Later this lecture, we will draw $\vec{H}$ in a <u>ferro</u>-magnet for contrast.

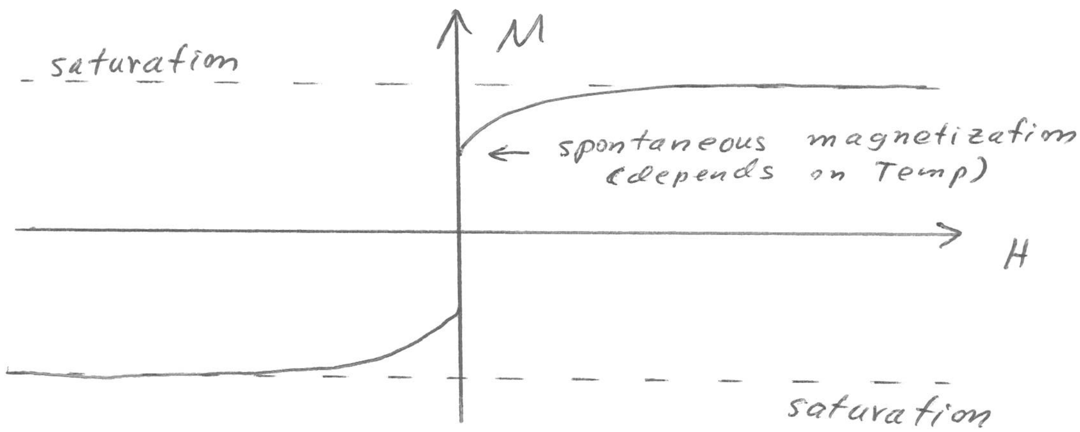
(iii) <u>Ferromagnetism</u>

Because of quantum Mechanical considerations (Pauli Exclusion Principle) atoms in a solid have a spin magnetic moment interaction between neighbors:

$$U_{12} = J\ \vec{m}_{(1)} \cdot \vec{m}_{(2)}$$

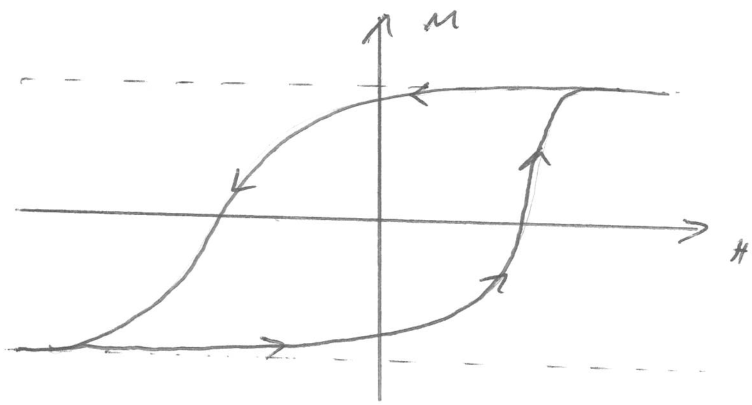
For some materials (depending on crystal lattice spacing, electronic wave function, etc.) J can be <u>negative</u>, and hence it is energetically favorable to have atomic magnetic dipole moments <u>aligned</u>. These are ferromagnetic materials $(\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow)$.

Below a certain temperature characteristic of the material (Curie temperature) a ferromagnetic sample would have a magnetization <u>even in the absence</u> of an external field $\vec{H}$. Turning on $\vec{H}$ will then further increase $\vec{M}$ by overcoming thermal effects until $\vec{M}$ saturates [when <u>all</u> the dipoles are aligned with the field, $\vec{M}$ can not grow any larger].



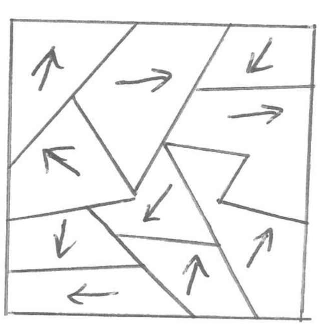
The spontaneous magnetization vanishes as $T \to T_{curie}$.

As the external field $\vec{H}$ changes direction, the sample "remembers" its previous configuration. This phenomenon is called <u>hysteresis</u>.



the arrows indicate <u>time</u>.

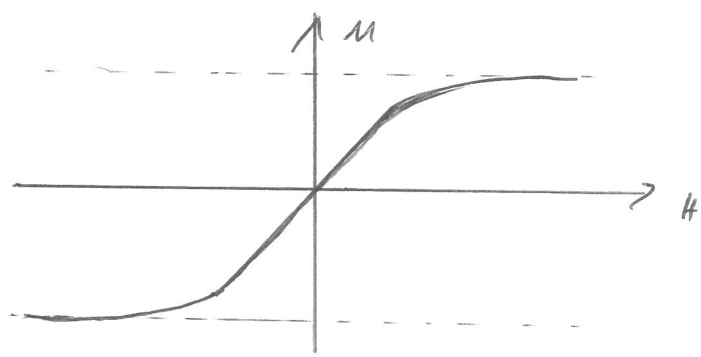
The origin of such a curve is in the domain structure of ferromagnetic materials. Each domain is like a small-scale magnet, but the individual magnetizations are oriented randomly. Within each domain, we have a magnetization curve like the one at the very top of the page.



Turning on $\vec{H}$ then aligns the domains. Because of frictional effects, decreasing $\vec{H}$ will not produce the same magnetization reversibly. At $\vec{H}=0$, there is still some residual magnetization.

In general, the relation between $\vec{M}$ and $\vec{H}$ is very difficult to describe mathematically, so we limit the discussion to 2 extreme cases:

① If a material has a very narrow hysteresis curve (essentially single-valued) then near $\vec{H}=0$ there is a linear region



<u>soft</u> ferro-magnetic material

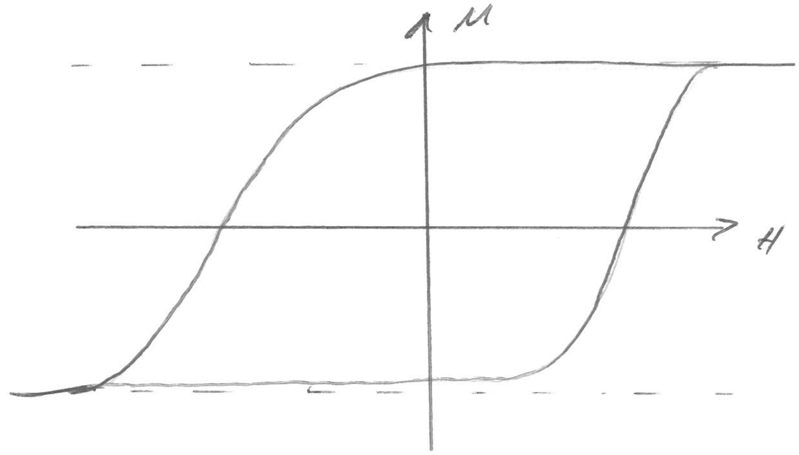
in which we can use $\vec{M} = \chi_f \vec{H}$. In this approximation, we must avoid the large fields $\vec{H}$ which would saturate the sample.

χ_f is the (soft) ferromagnetic susceptibility and it is very large compared to χ_p and χ_d.

χ_d (Bismuth) $= -14 \times 10^{-6}$ $\qquad$ χ_p (Manganese) $= 300 \times 10^{-6}$

$\chi_F \sim 50 \to 1000$ typically, several million times larger.

In the other extreme case, the material has a very wide hysteresis loop, so that for a restricted range in H, M is approximately <u>constant</u>:



<u>hard</u>
ferromagnetic material

The saturated values of M can be as large as 20,000 gauss = 2 Tesla.

When we are dealing with boundary-value problems, we will write

$$\vec{B} = \mu \vec{H} \quad \text{where} \quad \mu = 1 + 4\pi \chi$$

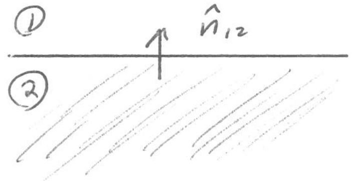
for χ_d - dia magnets
χ_p - para magnets
χ_f - <u>soft</u> ferro magnets

No such relation is possible for <u>hard</u> ferromagnets:

$\vec{M}$ independent of $\vec{H}$

Magnetic Boundary Conditions

Consider the interface between two regions of different magnetic permeability.

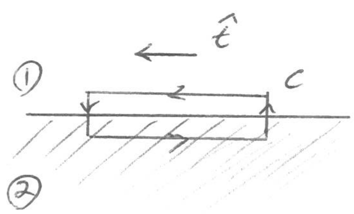


Then $\vec{\nabla} \cdot \vec{B} = 0$ implies (as in the electrostatic case) that $\hat{n}_{12} \cdot (\vec{B}_1 - \vec{B}_2) = 0$

(choose a gaussian pillbox that straddles the interface.)

The normal component of $\vec{B}$ is continuous across the interface.

The other magnetic Maxwell equation $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J}_{true}$ gives via Stoke's theorem:



$$\oint_C d\vec{l} \cdot \vec{H} = \frac{4\pi}{c} I_{enclosed}$$

Let the curve C shrink to zero in the direction perpendicular to the interface.

$$\hat{t} \cdot (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c} K$$

where $\hat{t}$ is a unit vector in the tangential direction (along the curve C) and K is the surface current density. We can also write this as

$$\hat{n}_{12} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c} \vec{K}$$

In the absence of true surface currents, the tangential components of $\vec{H}$ are continuous across the interface.

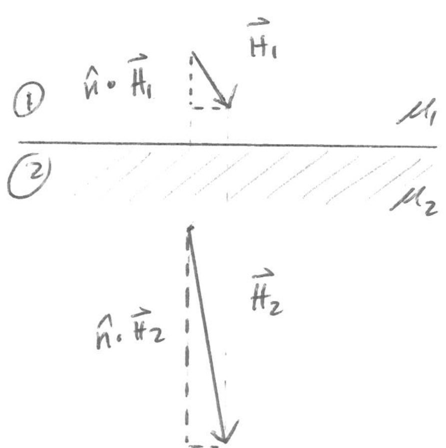
These interface conditions can be written in terms of the induction $\vec{B}$ or the intensity $\vec{H}$.

$$\hat{n}_{12} \cdot \vec{B}_1 = \hat{n}_{12} \cdot \vec{B}_2 \qquad \hat{n}_{12} \times \frac{\vec{B}_1}{\mu_1} = \hat{n}_{12} \times \frac{\vec{B}_2}{\mu_2}$$

$$\hat{n}_{12} \cdot \vec{H}_1 \mu_1 = \hat{n}_{12} \cdot \vec{H}_2 \mu_2 \qquad \hat{n}_{12} \times \vec{H}_1 = \hat{n}_{12} \times \vec{H}_2$$

If $\hat{n}_{12} \cdot \vec{H}_1 \neq 0$ and if $\mu_1 \gg \mu_2$ then

$$\hat{n}_{12} \cdot \vec{H}_2 \gg \hat{n} \cdot \vec{H}_1 \quad \text{while} \quad \hat{n}_{12} \times \vec{H}_1 = \hat{n}_{12} \times \vec{H}_2$$



Thus $\vec{H}_2$ is for all practical purposes normal to the interface. The $\vec{H}_2$ field lines behave like Electric field lines near a conductor. This analogy can be exploited in solving problems.

We now work through a few sample problems.

① Hard ferromagnets in the absence of currents

(i) Solution via the magnetic scalar potential

Since $\vec{J}=0$ we have $\vec{\nabla}\times\vec{H}=0 \Rightarrow \vec{H}=-\vec{\nabla}\Phi_m$.

$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{\nabla}\cdot(\vec{H}+4\pi\vec{M})=0 \Rightarrow \vec{\nabla}\cdot\vec{H}=-4\pi\vec{\nabla}\cdot\vec{M}$.

All together: $\nabla^2\Phi_m(\vec{r}) = 4\pi\,\vec{\nabla}\cdot\vec{M}(\vec{r})$

This is Poisson's equation with "magnetic charge density" $\rho_m(\vec{r}) = -4\pi\,\vec{\nabla}\cdot\vec{M}(\vec{r})$.

If there are no magnetic boundaries so that we only require $\Phi_m(\vec{r})\to 0$ as $r\to\infty$ then the solution is:

$$\Phi_m(\vec{r}) = -\int dV'\,\frac{\vec{\nabla}_{r'}\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Next, we integrate by parts and neglect the surface term.

$$\Phi_m(\vec{r}) = +\int dV'\,\vec{M}(\vec{r}')\cdot\vec{\nabla}_{r'}\,\frac{1}{|\vec{r}-\vec{r}'|}$$

Now change the gradient with respect to primed coordinates into unprimed coordinates (with a corresponding sign change) and pull the gradient out of the integral:

$$\Phi_m(\vec{r}) = -\vec{\nabla}\cdot\int dV'\,\frac{M(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

We now specialize to the case in which $\vec{M}$ is <u>uniform</u> inside the volume V.

$$\Phi_m(\vec{r}) = -\vec{\nabla}\cdot\left[\vec{M}\int dV' \frac{1}{|\vec{r}-\vec{r}'|}\right]$$

If V is a sphere of radius <u>a</u>, then for $r>a$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{4\pi}{2\ell+1}\left(\frac{r'}{r}\right)^{\ell} Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$$

and since $\int d^2\Omega'\, Y^*_{\ell m}(\theta',\varphi') = \delta_{\ell 0}\,\delta_{m0}\sqrt{4\pi}$

<u>$r>a$:</u> $\Phi_m(\vec{r}) = -\vec{\nabla}\cdot\frac{\vec{M}(Vol)}{r} = -\vec{m}\cdot\vec{\nabla}\frac{1}{r} = \frac{\vec{m}\cdot\vec{r}}{r^3}$

where $\vec{m} = \vec{M}\cdot$ Volume

Thus a uniformly magnetized sphere behaves <u>outside</u> like a point magnetic dipole at the origin with strength $\vec{m} = \vec{M}\cdot$(Volume of sphere)
This is exact, <u>not</u> an approximation.
For $r<a$ we have

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r_>} + \text{terms that involve } \ell>0 \text{ and which integrate to zero.}$$

$r_>$ is $\max(r, r')$ so

<u>$r<a$:</u>
$$\Phi_m(\vec{r}) = -\vec{\nabla}\cdot\vec{M}\left[\int_0^r 4\pi r'^2 dr'\,\frac{1}{r} + \int_r^a 4\pi r'^2 dr'\,\frac{1}{r'}\right]$$
$$= -\vec{\nabla}\cdot\vec{M}\left[\frac{4\pi}{3}r^2 + \frac{4\pi}{2}(a^2-r^2)\right] = +\vec{\nabla}\cdot\vec{M}\,\frac{4\pi r^2}{6}$$
$$= \frac{4\pi}{3}\vec{M}\cdot\vec{r}$$

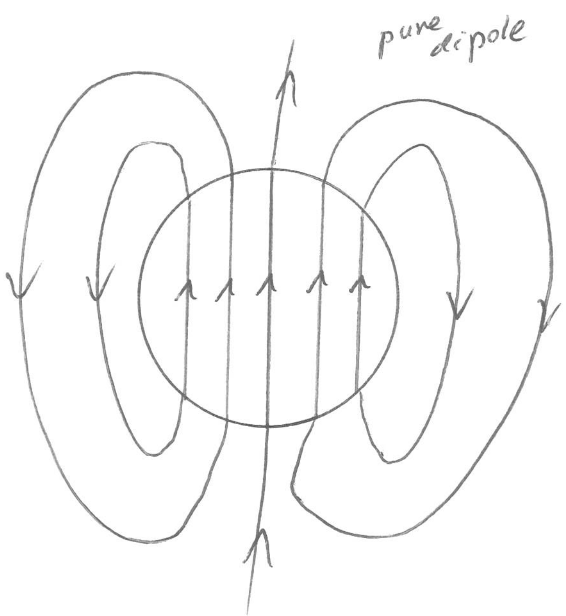
Thus inside we have $\vec{H} = -\vec{\nabla}\Phi_m(\vec{r}) = -\frac{4\pi}{3}\vec{M}$

while outside $\vec{H}(\vec{r}) = -\vec{\nabla}\Phi_m(\vec{r}) = \frac{4\pi a^3}{3}\,\frac{3(\vec{M}\cdot\vec{r})\vec{r} - r^2\vec{M}}{r^5}$

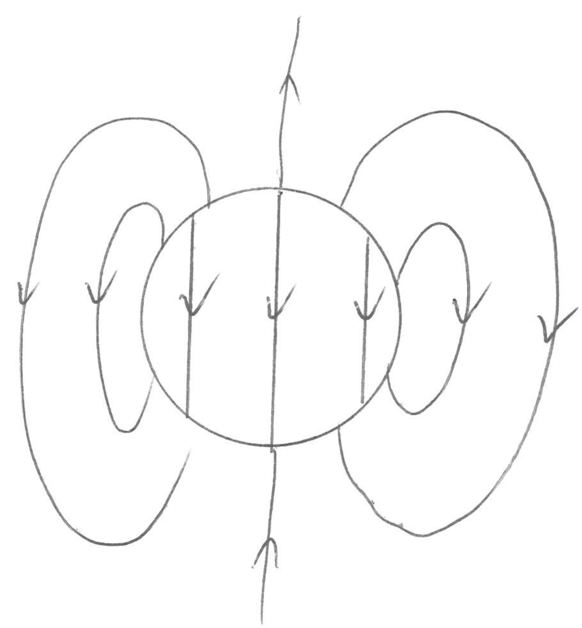
and since $\vec{B} = \vec{H} + 4\pi\vec{M}$ we get

outside: $\vec{B}(\vec{r}) = \frac{4\pi a^3}{3}\,\frac{3(\vec{M}\cdot\vec{r})\vec{r} - r^2\vec{M}}{r^5} = \vec{H}(\vec{r})$

inside: $\vec{B}(\vec{r}) = -\frac{4\pi}{3}\vec{M} + 4\pi\vec{M} = \frac{8\pi}{3}\vec{M}$



$\vec{B}$ field lines



$\vec{H}$ field lines

Note that the $\vec{B}$ field lines close on themselves since they have no sources or sinks: $\vec{\nabla}\cdot\vec{B} = 0$.

The $\vec{H}$ field lines do have a source

$\vec{\nabla}\cdot\vec{H} = -4\pi\,\vec{\nabla}\cdot\vec{M}$; it is the surface magnetization.

When $\vec{M}$ changes from a constant value inside to zero outside, $\vec{\nabla}\cdot\vec{M}$ is infinite (a delta function) at the surface, and this behaves like a surface monopole density of "magnetic charge."

(ii) Solution via vector potential for the same problem:

$$\vec{\nabla}\cdot\vec{B} = 0 \implies \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{H} + 4\pi\vec{M}) = 4\pi\,\vec{\nabla}\times\vec{M}$$

since $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}$ and $\vec{J} = 0$

To simplify the math, we choose to work in coulomb gauge : $\vec{\nabla}\cdot\vec{A}_0 = 0$

The solution is:

$$\vec{A}_0(\vec{r}) = \int dV' \, \frac{\vec{\nabla}_{r'}\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Again, we integrate by parts and neglect the surface term.

$$\vec{A}_0(\vec{r}) = \int dV' \, \vec{M}(\vec{r}') \times \vec{\nabla}_{r'} \frac{1}{|\vec{r}-\vec{r}'|}$$

Change the gradient from primed to unprimed:

$$\vec{A}_0(\vec{r}) = \vec{\nabla} \times \int dV' \, \frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The integral is exactly the same as before.

For example, for a uniformly magnetized sphere with $\vec{m} = \vec{M} \, \frac{4\pi}{3} a^3$

$$\vec{A}_0(\vec{r}) = \vec{\nabla} \times \frac{\vec{m}}{r} = \vec{\nabla}\left(\frac{1}{r}\right) \times \vec{m} = \frac{\vec{m} \times \vec{r}}{r^3} \qquad \underline{\underline{r > a}}$$

and

$$\vec{A}_0(\vec{r}) = -\vec{\nabla} \times \left(\vec{M} \frac{4\pi r^2}{6} \right) = \frac{4\pi}{3} \vec{M} \times \vec{r} \qquad \underline{\underline{r < a}}$$

This vector potential will lead to exactly the same $\vec{B}$ and $\vec{H}$ fields as the solution by scalar potential.

——— End Lecture #23 ———