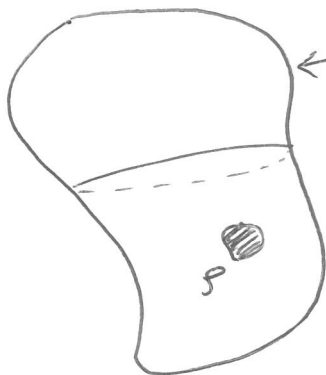


14 Sept 95

### The Dirichlet Problem



←  $\Phi(\vec{r}')$  specified on  $S$

$f(\vec{r})$  specified in  $V$

The solution  $\Phi(\vec{r})$  in  $V$  is uniquely determined.

### The Neumann Problem



←  $\hat{n}' \cdot \vec{\nabla}_{\vec{r}'} \Phi(\vec{r}')$  specified on  $S$

$f(\vec{r})$  specified in  $V$

The solution  $\Phi(\vec{r})$  in  $V$  is determined up to a constant.

### The Cauchy Problem



←  $\Phi(\vec{r}')$  specified on  $S$

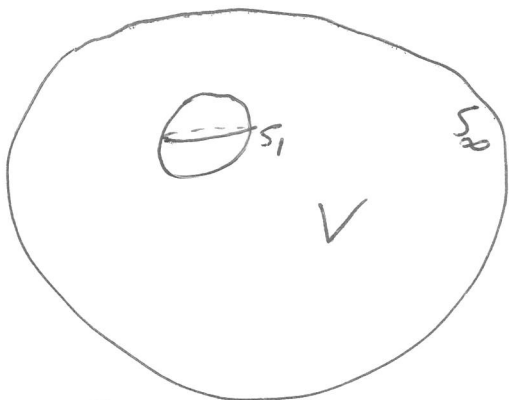
and  $\hat{n}' \cdot \vec{\nabla}_{\vec{r}'} \Phi(\vec{r}')$  specified on  $S$

$f(\vec{r})$  not specified in  $V$

Does a unique solution  $\Phi(\vec{r})$  exist in  $V$ ?

The surface  $\underline{S}$  which bounds the volume  $V$  of interest need not be simply-connected.

Consider the boundary given by a surface  $S_1$  and a surface at infinity  $S_\infty$ .



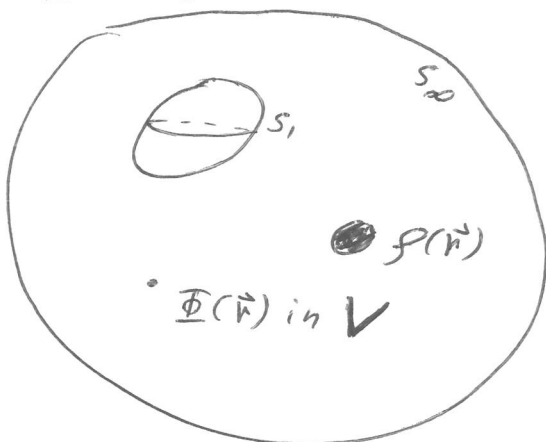
$$S = S_1 + S_\infty$$

The volume bound by this two-part surface is all space except that inside the bubble  $S_1$ .

The Dirichlet Problem then takes the form:

$\Phi(\vec{r})$  is specified on  $S_1$  and  $S_\infty$  ( $\Phi = 0$  on  $S_\infty$ )

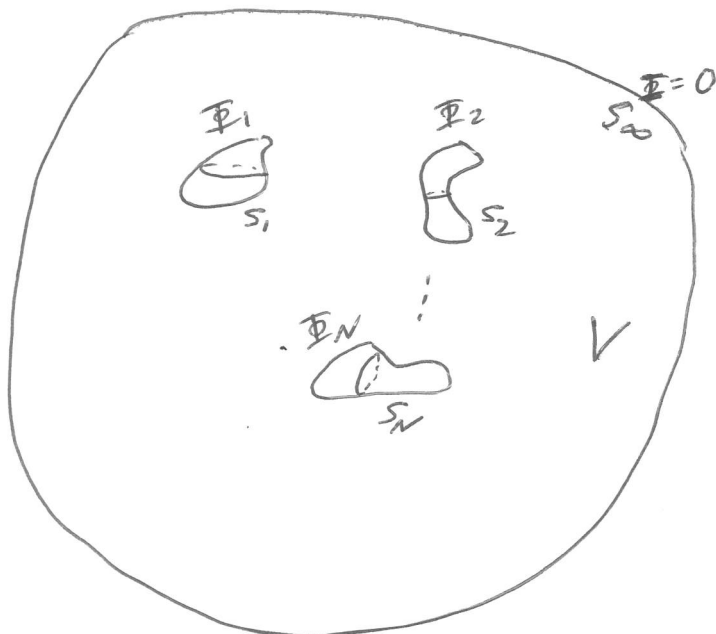
The charge density is known outside the bubble  $S_1$ . A unique solution  $\Phi(\vec{r})$  exists outside the bubble  $S_1$ .



Consider a system of  $N$  conductors. Remember from homework that there is no electric field in the bulk of the conductor, and if there are no charges in cavities of the conductor the field is zero there as well. This means that the conductors are equipotentials. The value of  $\Phi(\vec{r})$  is constant throughout a conductor, even if it is hollow.

Further suppose that the only charges around are at the surfaces of the  $N$  conductors. There is no charge density in the space between the conductors.

This is a Dirichlet Problem:



The surface  $S$  is not simply connected.

$$S = S_1 + S_2 + \dots + S_N + S_\infty$$

The bound volume is all space except the interiors of the  $N$  conductors

We will solve for the potential  $\Phi(\vec{r})$  in the space between the  $N$  conductors:

$$\Phi(\vec{r}) = \int_V dV' G_D(\vec{r}, \vec{r}') \rho(\vec{r}') - \oint_S \frac{dS'}{4\pi} \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}')$$

$\rho(\vec{r}')$  is zero in the volume  $V$  since charges only exist on the conducting surfaces.

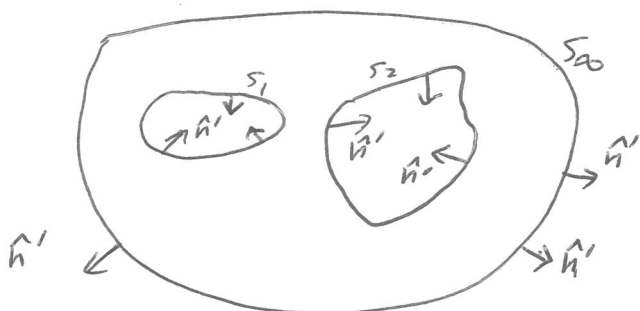
In the surface integrals,  $\Phi(\vec{r}')$  is constant on each surface  $S_i$  (a different constant for each surface!) and  $\Phi(\vec{r}') = 0$  on  $S_{\infty}$ .

$S_0, \dots$

$$\Phi(\vec{r}) = - \oint_{S_1 + S_2 + \dots + S_N} \frac{dS'}{4\pi} \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}')$$

$\hat{n}'$  is the outward pointing normal vector for the entire surface  $S = S_1 + S_2 + \dots + S_N + S_{\infty}$

$\hat{n}'$  actually points into the conductors.



Let  $\hat{n}'_j$  be the outward normal to surface  $S_j$ .

Then  $\hat{n}'_j = -\hat{n}'$

$$\Phi(\vec{r}) = + \sum_{j=1}^N \Phi_j \oint_{S_j} \frac{dS_j'}{4\pi} \hat{n}_j' \cdot \vec{\nabla}_{\vec{r}'} \sigma_0(\vec{r}, \vec{r}')$$


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Now let's go further. We just solved for the potential at all points in  $V$ . Since the conductors are not covered with a dipole moment density (just a monopole (charge) density), we know that the potential will be continuous right up to the conducting surfaces. Let's consider the potential on the  $i^{\text{th}}$  conductor.

$$\Phi_i = \Phi(\vec{r}) = \sum_{j=1}^N \Phi_j \oint_{S_j} \frac{dS_j'}{4\pi} \hat{n}_j' \cdot \vec{\nabla}_{\vec{r}'} \sigma_0(\vec{r}, \vec{r}')$$

for  $\vec{r}$  on  $S_i$  and  $\vec{r}'$  on  $S_j$

We also know that

$$-\hat{n}_i \cdot \vec{\nabla}_{\vec{r}} \Phi(\vec{r}) = +\hat{n}_i \cdot \vec{E}(\vec{r}) = 4\pi \sigma_i(\vec{r}) \text{ for } \vec{r} \text{ on } S_i$$

that is, the normal component of the electric field on the  $i^{\text{th}}$  surface gives the charge density on the  $i^{\text{th}}$  surface.

$$\begin{aligned}
 -\hat{n}_i \cdot \vec{\nabla}_r \Phi(\vec{r}) &= +\hat{n}_i \cdot \vec{E}(\vec{r}) = 4\pi \sigma_i(\vec{r}) \\
 &= -\sum_{j=1}^N \Phi_j \oint_{S_j} \frac{dS_j'}{4\pi} \hat{n}_i \cdot \vec{\nabla}_r \hat{n}_j' \cdot \vec{\nabla}_{r'} G_D(\vec{r}, \vec{r}')
 \end{aligned}$$

If we integrate the surface charge density  $\sigma_i$ , we get the charge on the  $i^{\text{th}}$  conductor  $q_i$ .

$$\begin{aligned}
 q_i &= \oint_{S_i} dS \sigma_i = -\sum_{j=1}^N \oint_{S_i} \frac{dS_i}{4\pi} \oint_{S_j} \frac{dS_j'}{4\pi} \hat{n}_i \cdot \vec{\nabla}_r \hat{n}_j' \cdot \vec{\nabla}_{r'} G_D(\vec{r}, \vec{r}') \Phi_j \\
 &\equiv \sum_{j=1}^N C_{ij} \Phi_j
 \end{aligned}$$

where

$$C_{ij} = -\oint_{S_i} \frac{dS_i}{4\pi} \oint_{S_j} \frac{dS_j'}{4\pi} \hat{n}_i \cdot \vec{\nabla}_r \hat{n}_j' \cdot \vec{\nabla}_{r'} G_D(\vec{r}, \vec{r}')$$

This looks symmetric in  $i$  and  $j$ . Prove this.

Remember that  $G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r})$ .

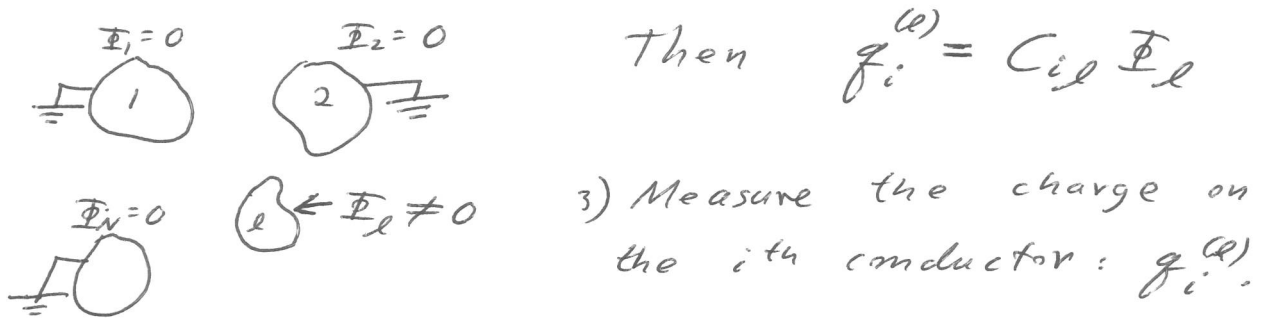
This is a capacitance matrix.

The  $C_{ii}$  are called capacities of conductor  $i$ .

The  $C_{ij}$  with ( $i \neq j$ ) are called coefficients of induction between conductor  $i$  and conductor  $j$ .

How to measure the elements of the capacitance matrix experimentally:

- 1) Ground all the conductors, except one, say the  $l^{\text{th}}$ .
- 2) Hold the  $l^{\text{th}}$  conductor at potential  $\Phi_l$



Then  $q_i^{(l)} = C_{il} \Phi_l$

- 3) Measure the charge on the  $i^{\text{th}}$  conductor:  $q_i^{(l)}$ .

The matrix element is  $C_{il} = \frac{q_i^{(l)}}{\Phi_l}$

- 4) Repeat for all  $N$  conductors in turn.

Notice that once you have determined

$C_{as}$  by measuring  $q_a^{(s)}$  while conductor #  $S$  is held at  $\Phi_s$ , you know  $C_{sa} = C_{as}$ .

So there is no need to measure the charge on conductor #  $S$  while conductor #  $2$  is held at  $\Phi_2$ . (Unless you want to verify the symmetry!)

The matrix elements  $C_{ij}$  depend on:

- the shape of the conductors,
- the relative position and orientation of the conductors,
- the dielectric medium if any is present.

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A very powerful problem-solving tool is  
Green's Reciprocation Theorem

Consider two distinct charge distributions.  
First point charges, then later charge densities.

set A :  $\{q_i^{(A)}\}$   $i = 1, \dots, N_A$  symbol  $\circ$  at  $\vec{r}_i$

set B :  $\{q_j^{(B)}\}$   $j = 1, \dots, N_B$  symbol  $\times$  at  $\vec{r}_j$

The charges can be intertwined:

$\circ \times \circ \times \circ$   
 $\circ \circ \times \times \circ \circ$   
 $\times \times$

but no charge in set A can be  
in set B at the same time.

The potential energy of the configuration  
due to the interaction of set A and set B is

$$U_{AB} = \sum_i \sum_j \frac{q_i^{(A)} q_j^{(B)}}{|\vec{r}_i - \vec{r}_j|}$$



We can write this in two ways:

$$U_{AB} = \sum_i q_i^{(A)} \left( \sum_j \frac{q_j^{(B)}}{|\vec{r}_i - \vec{r}_j|} \right) = \sum_i q_i^{(A)} \Phi^{(B)}(\vec{r}_i)$$

or

$$U_{AB} = \sum_j q_j^{(B)} \left( \sum_i \frac{q_i^{(A)}}{|\vec{r}_j - \vec{r}_i|} \right) = \sum_j q_j^{(B)} \Phi^{(A)}(\vec{r}_j)$$

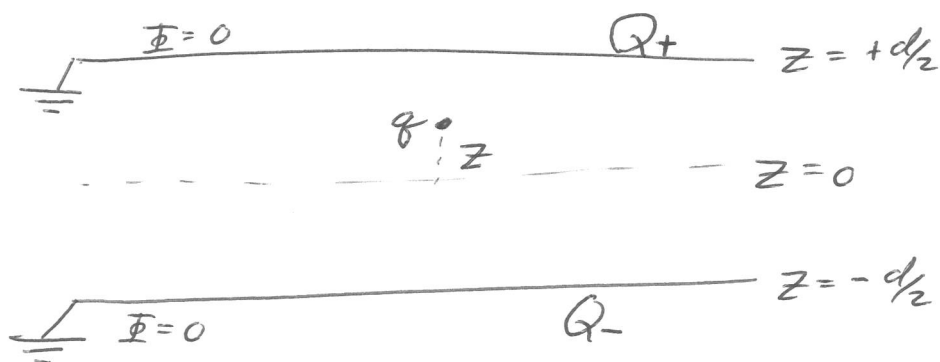
where  $\Phi^{(A)}(\vec{r}_j)$  is the potential due to all the charges in set A, evaluated at the position  $\vec{r}_j$  of a charge in set B,  $q_j^{(B)}$ .

The equality of these two results is Green's Reciprocation Theorem:

$$\sum_{i=1}^{N_A} q_i^{(A)} \Phi^{(B)}(\vec{r}_i) = \sum_{j=1}^{N_B} q_j^{(B)} \Phi^{(A)}(\vec{r}_j)$$

Let's solve a problem with G.R.T.

Consider two grounded infinite parallel conducting planes separated by a distance  $d$ . A point charge sits somewhere (not halfway) between the plates. What is the charge induced on each plate?



First, we can use Gauss' Law to determine the sum  $Q_+ + Q_-$ . Consider a surface that surrounds both plates. Since  $\vec{E} = 0$  outside the plates, the total enclosed charge must be zero:

$$q + Q_+ + Q_- = 0$$

So 
$$Q_+ + Q_- = -q$$

Now we will use Green's Reciprocation Theorem.

For the charges in set A, choose:  $q$ ,  $Q_+$ , and  $Q_-$ .  
That's everything! Notice that  $Q_+$  and  $Q_-$   
are not distributed evenly over the plates.

For the charges in set B, choose some that are  
not in the problem. They are a device to get  
the answer we seek. Choose a uniform  
charge density  $\sigma$  on the upper plate and  
charge density  $-\sigma$  on the lower plate.

What is the potential due to the B charges?

$$\Phi_z^{(B)} = \overset{+?}{-} 4\pi\sigma z + \Phi_0 \quad \text{between the plates}$$

What is the potential due to the A charges  
at the positions of the B charges? That is,  
what is  $\Phi^{(A)}(d/2)$  and  $\Phi^{(A)}(-d/2)$ ? Since  
the plates are grounded these are both zero.

$$\sum_j q_j^{(B)} \Phi^{(A)}(\vec{r}_j) = 0 = \sum_i q_i^{(A)} \Phi^{(B)}(\vec{r}_i)$$

so

$$0 = Q_+ \left[ -4\pi\sigma \frac{d}{2} + \Phi_0 \right] + Q_- \left[ -4\pi\sigma \left(-\frac{d}{2}\right) + \Phi_0 \right] + q \left[ -4\pi\sigma z + \Phi_0 \right]$$

We have two equations in two unknowns,  
which enables us to solve for  $Q_+$  and  $Q_-$ .

$$\text{Find: } Q_{\pm} = -\frac{q}{2} \left( 1 \pm \frac{2z}{d} \right)$$

check the symmetric case: when  $z=0$ , the  
point charge is exactly midway between the  
plates and this should induce charges of  
 $-\frac{q}{2}$  on each plate. ✓

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The other way to solve this problem is to  
find a solution  $\Phi(\vec{r})$  for the potential  
between the plates that satisfies Poisson's  
Equation:

$$\nabla^2 \Phi(\vec{r}) = -4\pi \rho(\vec{r}) = -4\pi q \delta^3(\vec{r} - \vec{r}_q)$$

and the boundary conditions:

$$\Phi(z = \pm d/2) = 0$$

Then calculate  $\vec{E} = -\vec{\nabla} \Phi$  and at the plates  
find  $\hat{n} \cdot \vec{E} = 4\pi \sigma_{\pm}(x, y)$ . Then find the

charge

$$Q_{\pm} = \int_{-a}^{+a} dx \int_{-a}^{+a} dy \sigma_{\pm}(x, y).$$

This is a lot of work!

A clever choice of the set B charges can lead to a simpler solution with G, R, T.

Of course, this trick only works for relatively simple geometries. The brute-force integration method always works.

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And now a word about Green functions:

$$\nabla_r^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi \delta^3(\vec{r}-\vec{r}')$$

$$\nabla_r^2 G_D(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r}-\vec{r}')$$

$$\nabla_r^2 G_N(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r}-\vec{r}')$$

What do the Green functions  $G_D$  and  $G_N$  look like?

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F_D(\vec{r}, \vec{r}')$$

$$G_N(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F_N(\vec{r}, \vec{r}')$$

where  $\nabla_r^2 F_D(\vec{r}, \vec{r}') = 0 = \nabla_r^2 F_N(\vec{r}, \vec{r}')$

$F_D$  ensures that  $G_D(\vec{r}, \vec{r}')$  vanishes on  $S$ .  $\frac{1}{|\vec{r}-\vec{r}'|}$  does not.

$F_N$  ensures that  $\vec{\nabla}[G_N(\vec{r}, \vec{r}')] \cdot \hat{n} = -\frac{4\pi}{S}$  on  $S$ .  $\frac{1}{|\vec{r}-\vec{r}'|}$  does not.