E) The Method of Series Expansion

In the following, we will describe a method for solving Laplace's equation

$$\nabla^2 \Phi = 0$$

which involves the use of orthogonal functions.

A set of functions $u_1(\xi), u_2(\xi), \ldots u_N(\xi)$ is called orthogonal on the interval $[a,b]$ if

$$\int_a^b d\xi \; u_n^*(\xi) \; u_m(\xi) = k_n \delta_{nm}$$

where $u_n^*(\xi)$ is the complex conjugate of $u_n(\xi)$.

If $k_n = 1$, the the functions are called orthonormal. Some functions (like plane waves) cannot be normalized, but they can still be orthogonal.

Think of the integral above as a "dot-product." There is a strong analogy between orthogonal functions and orthogonal vectors, We seek an approximation to the function $f(\xi)$ as a series of orthogonal functions:

$$f_N(\xi) = \sum_{n=1}^{N} c_n u_n(\xi)$$

The error in such an approximation is <u>defined</u> to be

$$E_N(c_1, c_2, \ldots c_N) = \int_a^b d\xi \, |f(\xi) - f_N(\xi)|^2$$

The choice of expansion coefficients c_n (which are complex in general) that minimizes the error for fixed N is

$$c_n = \int_a^b d\xi \, u_n^*(\xi) f(\xi)$$

A set of functions $\{u_n(\xi)\}$ is said to be <u>complete</u> if

$$\lim_{N \to 0} E_N(c_1, c_2, \ldots c_N) = 0$$

where the c_n are chosen as above and $f(\xi)$ is arbitrary (but $\int_a^b d\xi \, |f(\xi)|^2$ must exist).

The sequence of approximations $\{f_N(\xi)\}$ is said to converge in the mean to the function $f(\xi)$.

In physical situations, the function $f(\xi)$ will be related to the electrostatic potential and will be sufficiently smooth (e.g. continuous) that the convergence will be even stronger than "in the mean." The series $\sum_{n=1}^{\infty} c_n u_n(\xi)$ will converge uniformly to $f(\xi)$ at points for which $f(\xi)$ is continuous and will converge to

$$\frac{1}{2}\left[f(\xi+\epsilon)+f(\xi-\epsilon)\right]_{\epsilon\to 0^+} \quad \text{for points at}$$

which $f(\xi)$ is discontinuous.

As examples, we consider trigonometric functions (sines and cosines or complex exponentials) which are used in the Fourier series.

Consider a function which is periodic with period L,

RESERVE $f(x+L) = f(x)$

1) Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} \underbrace{g_n}_{\substack{\text{expansion}\\ \text{coefficients}}} \underbrace{e^{i\frac{2\pi n x}{L}}}_{u_n(x)}$$

$$g_n = \frac{1}{L}\int_{-L/2}^{+L/2} dx\, \underbrace{e^{-i\frac{2\pi n x}{L}}}_{u_n^*(x)} f(x)$$

2) Real Fourier Series

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

$$A_0 = \frac{1}{L}\int_{-L/2}^{+L/2} dx\, f(x) = \langle f \rangle \quad \text{average value}$$

$$\left.\begin{aligned} A_n &= \frac{2}{L}\int_{-L/2}^{+L/2} dx\, \cos\left(\frac{2\pi n x}{L}\right) f(x) \\ B_n &= \frac{2}{L}\int_{-L/2}^{+L/2} dx\, \sin\left(\frac{2\pi n x}{L}\right) f(x) \end{aligned}\right\} \quad n > 0$$

Separation of Variables

Laplace's equation can be expressed in general orthogonal curvilinear coordinates as

$$\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial q_i}\left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial}{\partial q_i} \Phi \right) = 0 .$$

where in general h_i is a function of all of the generalized coordinates $h_i = h_i(q_1, q_2, q_3)$.

In cartesian coordinates: $h_i = 1$, $q_i = x_i$ so

$$\nabla^2 \Phi = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = 0$$

There are 11 orthogonal curvilinear coordinate systems in which the solution to Laplace's Equation factorizes into a product of three functions, each of which is <u>separately</u> a function of only one variable.

$$\Phi(q_1, q_2, q_3) = f_1(q_1)\, f_2(q_2)\, F_3(q_3)$$

They are:

1) Cartesian
2) Cylindrical Polar
3) Spherical Polar
4) Con Focal Ellipsoidal
5) Prolate Spheroidal
6) Oblate Spheroidal
7) Elliptic Cylindrical
8) Conical
9) Parabolic Cylindrical
10) Parabolic
11) Confocal Paraboloidal

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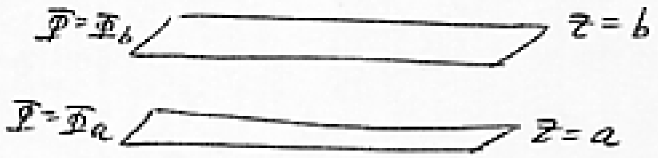
Some simple examples:

First let's consider a problem with plane boundaries. The geometry will be easiest to analyse in Cartesian coordinates.

i) boundary conditions that depend on one coordinate (say z) only: Suppose we have two planes maintained at constant potentials



The potential which is a solution to Laplace's equation between the planes can only be a function of z

$$\Phi = \Phi(z)$$

$$\nabla^2 \Phi = \frac{\partial^2}{\partial z^2} \Phi(z) = 0$$

Solution: $\Phi(z) = c_1 z + c_2$

The integration constants c_1 and c_2 are determined from the boundary conditions:

$$\Phi(z=a) = c_1 a + c_2 = \Phi_a$$
$$\Phi(z=b) = c_1 b + c_2 = \Phi_b$$

ii) boundary conditions that depend on 2 coordinates;

Consider two parallel grounded half planes $x=0$ $(y>0)$ and $x=a$ $(y>0)$ on which $\Phi=0$. These half planes are connected by an infinite strip $(0 \leq x \leq a)$ $y=0$ on which the potential is $\Phi_0(x)$. The other boundary surface is at $y=\infty$ where $\Phi=0$.

These boundary conditions are independent of z, therefore Φ will also be independent of z. $\Phi = \Phi(x,y)$

We assume a factorized solution

$$\Phi(x,y) = X(x)\,Y(y)$$

then Laplace's equation is

$$\nabla^2 \Phi(x,y) = X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

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The last equation says that a function of x equals a function of y. Since x and y are independent variables we must have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \begin{array}{c}\text{separation}\\\text{constant}\end{array} = -\alpha^2$$

where we have chosen to call the separation constant $-\alpha^2$ with α real and positive in anticipation of future developments.

Note that α cannot be zero:

$$\frac{X''(x)}{X(x)} = 0 \quad \Rightarrow \quad X(x) = c_1 x + c_2$$

but $X(x)$ must vanish at $x = 0$ and $x = a$, this means that $c_1 = 0 = c_2$ so the potential vanishes everywhere.

We are left with two <u>separate</u> differential equations:

$$X''(x) + \alpha^2 X(x) = 0$$

$$Y''(y) - \alpha^2 Y(y) = 0$$

which have general solutions:

$$X(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$$

$$Y(y) = d_1 e^{-\alpha y} + d_2 e^{+\alpha y}$$

The boundary condition at $x=0$ is

$$X(0) = 0 = c_2$$

and the boundary condition at $y=\infty$ is

$$Y(\infty) = 0 = d_2 \qquad \text{(remember } \alpha > 0\text{)}$$

The requirement at $x=a$ is

$$X(a) = 0 = c_1 \sin(\alpha a)$$

we do not want $c_1 = 0$ because then $\Phi = 0$ everywhere,

so $\sin(\alpha a) = 0 \Rightarrow \alpha a = n\pi$ with $n = 1, 2, 3, \ldots$

we do not want $n=0$ because $\Phi = 0$ also.

The solution so far looks like:

$$\Phi(x,y) = c_1 d_1 \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

which have general solutions:

$$X(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$$

$$Y(y) = d_1 e^{-\alpha y} + d_2 e^{+\alpha y}$$

The boundary condition at $x=0$ is

$$X(0) = 0 = c_2$$

and the boundary condition at $y=\infty$ is

$$Y(\infty) = 0 = d_2 \qquad \text{(remember } \alpha > 0\text{)}$$

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$$X(a) = 0 = c_1 \sin(\alpha a)$$

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we do not want $n=0$ because $\Phi = 0$ also,

The solution so far looks like:

$$\Phi(x,y) = c_1 d_1 \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

Now we combine the two constants c_1 and d_1 into a single constant A_n and realize that there are solutions for $n = 1, 2, 3, \ldots$

The most general solution is a linear combination of solutions:

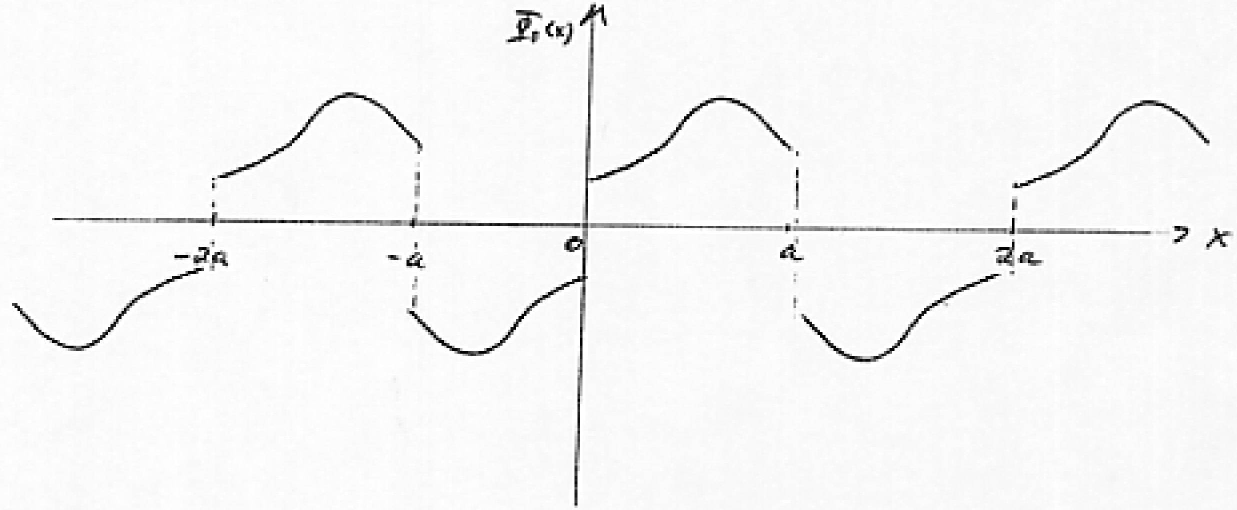
$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

This satisfies laplace's equation in the channel and all of the boundary conditions except the one at $y=0$. We now show how to choose A_n above to satisfy the last boundary condition:

$$\Phi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) = \Phi_0(x)$$

Notice that $\Phi_0(x)$ is only defined in $0 \le x \le a$. Furthermore, we can't ask for the value of the potential outside the volume V, that is, outside the channel. We can extend $\Phi_0(x)$ beyond the channel any way we please.

Suppose that we continue the function $\Phi_o(x)$ outside the region of physical interest by making $\Phi_o(x)$ odd and periodic of period $\underline{2a}$.



Now $\Phi_o(x)$ for x in $(-\infty, +\infty)$ has two properties:

periodic: $\Phi_o(x) = \Phi_o(x+2a)$

odd: $\Phi_o(x) = -\Phi_o(-x)$

A Fourier Series for $\Phi_o(x)$ would be

$$\Phi_o(x) = \cancel{a_o}^{\,0} + \sum_{k=1}^{\infty}\left[\cancel{a_k}^{\,0}\cos\left(\frac{2k\pi x}{2a}\right) + b_k \sin\left(\frac{2k\pi x}{2a}\right)\right]$$

The fact that Φ_o is odd means that $a_o, a_k = 0$.

$$b_k = \frac{2}{2a}\int_{-a}^{+a} dx\, \Phi_o(x)\, \sin\left(\frac{2k\pi x}{2a}\right)$$

or
$$b_K = \frac{1}{a}\int_{-a}^{+a} dx\, \Phi_0(x) \sin\left(\frac{k\pi x}{a}\right)$$

or since both $\Phi(x)$ and $\sin\left(\frac{k\pi x}{a}\right)$ are odd:

$$b_K = \frac{2}{a}\int_{0}^{a} dx\, \Phi_0(x) \sin\left(\frac{k\pi x}{a}\right)$$

we can double the result of integrating over half the period.

Now know how to choose the A_n in our general solution for the potential $\Phi(x,y)$.

$$A_n = b_n = \frac{2}{a}\int_0^a dx\, \Phi_0(x) \sin\left(\frac{n\pi x}{a}\right)$$

then

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

matches all the boundary conditions, including the one at $y=0$.

One last note: If $\Phi_0(0) \neq 0$ and $\Phi_0(a) \neq 0$ then there is a discontinuity in the boundary conditions, since $\Phi = 0$ on $x=0$ and $x=a$ planes.

Our solution for the interior of the channel in Fourier series is continuous, however, since it is made from continuous functions.

Our solution converges to

$$\Phi(0,0) = \frac{1}{2}\left[\Phi_0(0) + 0\right]$$

$$\Phi(a,0) = \frac{1}{2}\left[\Phi_0(a) + 0\right]$$

The separation constant $-\alpha^2$ was chosen this way because $\Phi(x,y)$ was required to vanish at 2 x-coordinates $x=0$ and $x=a$. A combination of exponentials can vanish in at most one location. Thus we needed sines + cosines on the x-axis.

Another way: sines + cosines are complete, real exponentials (not complex) are not complete. We need a complete set to approximate $\Phi_0(x)$.

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End Lecture # 8