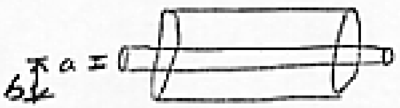
10 October 95

2) Suppose the region of interest is <u>between</u> two full infinite cylinders. The solution $\Phi(\rho, \varphi)$ must still be



periodic in $\varphi \Rightarrow c_1 = 0 = c_1'' = c_2''$
$\Rightarrow \alpha = n$

This time, the $\rho = 0$ axis is <u>excluded</u>, so the most general solution is

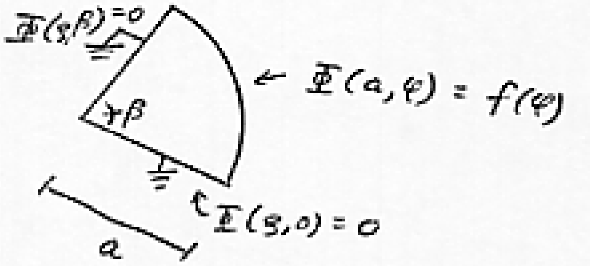
$$\Phi(\rho, \varphi) = A_0 + A_0' \ln\rho + \sum_{n=1}^{\infty} \rho^n \left[A_n \cos(n\varphi) + B_n \sin(n\varphi) \right]$$
$$+ \sum_{n=1}^{\infty} \rho^{-n} \left[A_n' \cos(n\varphi) + B_n' \sin(n\varphi) \right]$$

The expansion coefficients $A_0, A_0', A_n, B_n, A_n', B_n'$ are determined by Fourier analyzing the two bounding surfaces $\rho = a$, $\rho = b$. Dirichlet boundary conditions might be:

$$\begin{cases} \Phi(a, \varphi) = f_1(\varphi) \\ \Phi(b, \varphi) = f_2(\varphi) \end{cases}$$

RESERVE

3) A <u>sector</u> of an infinitely long circular cylinder with the following Dirichlet boundary conditions:



- We want a <u>complete</u> set of functions of φ to reproduce $f(\varphi)$ on the boundary. $\Rightarrow$ type (2) solution

The boundary conditions imply: $c_2'=0$ and $\alpha = \frac{n\pi}{\beta}$

$$\Phi(\rho,\varphi) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi\varphi}{\beta}\right) \rho^{\frac{n\pi}{\beta}}$$

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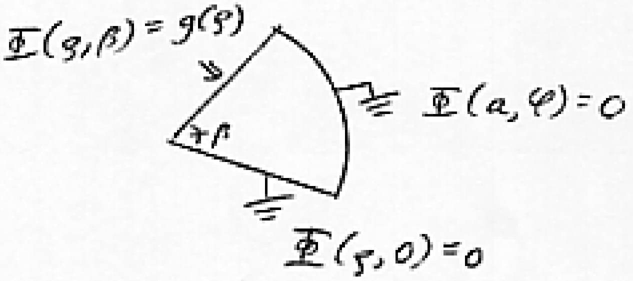
4) A sector, but with different boundary conditions:



— Now we want a <u>complete</u> set of functions of ρ to reproduce $g(\rho)$ on the boundary $\Rightarrow$ type (3) solution

These boundary conditions imply: $C_2''=0$, $D_2''=0$

$$\Phi(\rho,\varphi) = \int_0^{\infty} d\alpha\, B(\alpha) \sinh(\alpha\varphi) \sin\left[\alpha \ln\left(\frac{\rho}{a}\right)\right]$$ RESERVE

This is a Fourier <u>transform</u> rather than a Fourier series because there is no longer a restriction that α must be integral. α assumes <u>all</u> real values.

iii) Boundary Conditions depend on all 3 coordinates:

Look for factorizable solutions $\Phi(\rho,\varphi,z) = R(\rho)F(\varphi)Z(z)$.

$$\frac{\nabla^2\Phi}{\Phi} = 0 = \underbrace{\frac{(\rho R')'}{\rho R} + \frac{1}{\rho^2}\frac{F''}{F}}_{f(\rho,\varphi)} + \underbrace{\frac{Z''}{Z}}_{g(z)} = 0$$

At first glance it does not appear that we have succeeded in separating the variables since ρ and φ are still entangled, but notice that the first two terms together are a function of ρ and φ alone and the last term is a function of z alone. This can only hold true if both functions are <u>constant</u>. Call the <u>first</u> separation constant C.

$$\frac{(\rho R')'}{\rho R} + \frac{1}{\rho^2}\frac{F''}{F} = C \qquad\qquad \frac{Z''}{Z} = -C$$

The first equation can be rewritten as:

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$$\underbrace{\frac{\rho(\rho R')'}{R} - \rho^2 C}_{h_1(\rho)} + \underbrace{\frac{F''}{F}}_{h_2(\varphi)} = 0$$

The first two terms form a function of ρ alone and the third term is a function of φ alone. Call the <u>second</u> separation constant $\underline{K}$. The fully separated equations are:

$$Z'' + CZ = 0$$

$$F'' + \underline{K}F = 0$$

$$\frac{\rho(\rho R')'}{R} - \rho^2 C = \underline{K}$$

In general, the two constants of separation are arbitrary real numbers — positive, negative, or zero. There are too many special case geometries to deal with in detail, so we will confine our discussion to one particular problem — a full circular cylinder of radius <u>a</u> and length <u>L</u>.

Since the <u>full</u> range of φ is included in the problem, the solution to Laplace's equation, $\Phi(\rho, \varphi, z)$, must be periodic in φ with period 2π.

For this special case $\underline{K} = n^2$ where $n = 0, \pm 1, \pm 2, \ldots$

$$F''_{(\varphi)} = -n^2 F(\varphi) \Rightarrow F(\varphi) = a_1 \cos(n\varphi) + a_2 \sin(n\varphi)$$

$$\underset{\sim}{\text{or}} = a_3 e^{in\varphi}$$

RESERVE

The remaining differential equations are:

$$Z'' + CZ = 0 \quad \text{and} \quad \frac{\rho(\rho R')'}{R} - \rho^2 C = n^2$$

There are still three sub-cases to consider: the first separation constant, C, can be positive, negative or zero:

1) $\underline{C = 0}$

$$Z'' = 0 \implies Z(z) = b_1 z + b_2$$

$$\rho(\rho R')' - n^2 R = 0 \implies \begin{cases} n = 0, & R(\rho) = d_1 \ln\rho + d_2 \\ n \neq 0, & R(\rho) = c_1 \rho^{|n|} + c_2 \rho^{-|n|} \end{cases}$$

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2) $\underline{C = -k^2}$ $\quad k$ real, positive

$$Z'' - k^2 Z = 0 \rightarrow Z(z) = b_1' e^{kz} + b_2' e^{-kz}$$

$$\text{or} = \beta_1' \sinh(kz) + \beta_2' \cosh(kz)$$

$$\frac{1}{\rho}(\rho R')' + \left(k^2 - \frac{n^2}{\rho^2}\right)R = 0 \rightarrow R(\rho) = d_1' J_n(k\rho) + d_2' N_n(k\rho)$$

where $J_n(u)$ is the Bessel function of integer order $\underline{n}$.

and $N_n(u)$ is the Neumann function of integer order $\underline{n}$.

J_n and N_n are <u>complete.</u> Any function of ρ can be expanded in J_n and N_n.

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$J_n(u)$ is also called the Bessel function of the first type. $N_n(u)$ is also called the Bessel function of the second type, or the Weber function and the symbol is sometimes written $Y_n(u)$. In Mathematica, they are denoted BesselJ and BesselY, respectively.

These functions are defined for negative integer order by:

$$J_{-n}(u) \equiv (-1)^n J_n(u)$$

$$N_{-n}(u) \equiv (-1)^n N_n(u)$$

$J_n(u)$ and $N_n(u)$ are oscillatory functions. Think of them as cylindrical coordinate versions of sines and cosines. They each have an infinite number of zeroes.

$$J_n(u_{ns}) = 0 \qquad N_n(\bar{u}_{ns}) = 0 \qquad s = 1, 2, \ldots$$

The zeroes are labeled by s. These zeroes are tabulated or available from Mathematica.

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The <u>derivatives</u> of these functions also oscillate and have an infinite number of zeroes,

$$J_n'(u_{ns}') = 0 \qquad N_n'(\bar{u}_{ns}') = 0 \qquad s = 1, 2, \ldots$$

Asymptotically

$$J_n(u) \xrightarrow[u\to\infty]{} \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{\pi n}{2} - \frac{\pi}{4}\right) \qquad n \geq 0$$

$$N_n(u) \xrightarrow[u\to\infty]{} \sqrt{\frac{2}{\pi u}} \sin\left(u - \frac{\pi n}{2} - \frac{\pi}{4}\right) \qquad n \geq 0$$

The Bessel's functions of the <u>first</u> type are well-behaved at the origin:

$$J_n(u) \xrightarrow[u\to 0]{} \frac{1}{n!} \frac{u^n}{2^n} \qquad n \geq 0$$

The Neumann functions <u>diverge</u> at the origin

$$N_0(u) \xrightarrow[u\to 0]{} \frac{2}{\pi} \ln\left(\frac{u}{2}\right) + \frac{2}{\pi} \gamma_E$$

$$N_n(u) \xrightarrow[u\to 0]{} -\frac{1}{\pi} (n-1)! \left(\frac{2}{u}\right)^n \qquad n \geq 1$$

Hence, when the $\rho = 0$ axis is included in the physical region, we must <u>exclude</u> the Neumann functions, $N_n(u)$.

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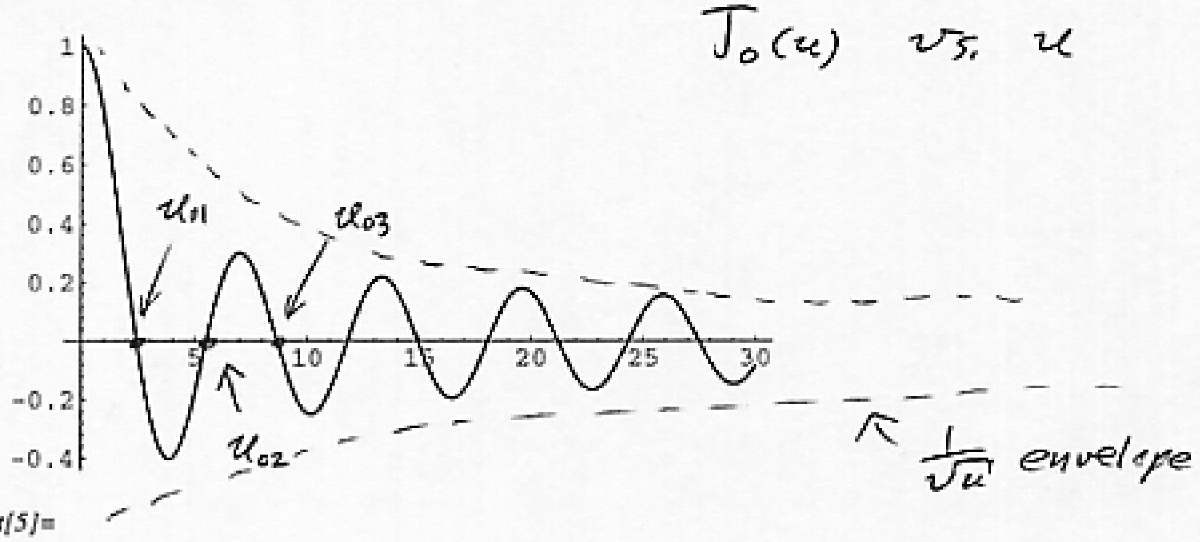
In[5]:=

```
Plot[ BesselJ[0,u],{u,0,30}]
```

$J_0(u)$ vs. u



Out[5]=

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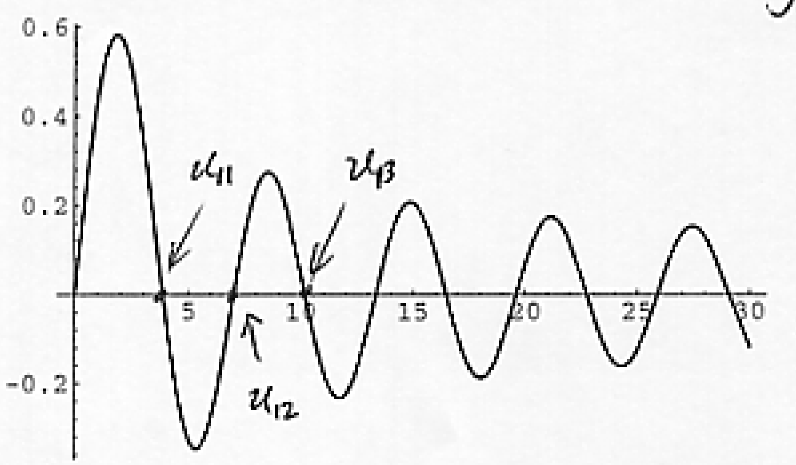
In[6]:=

```
Plot[ BesselJ[1,u],{u,0,30}]
```

$J_1(u)$ vs. u



Out[6]=

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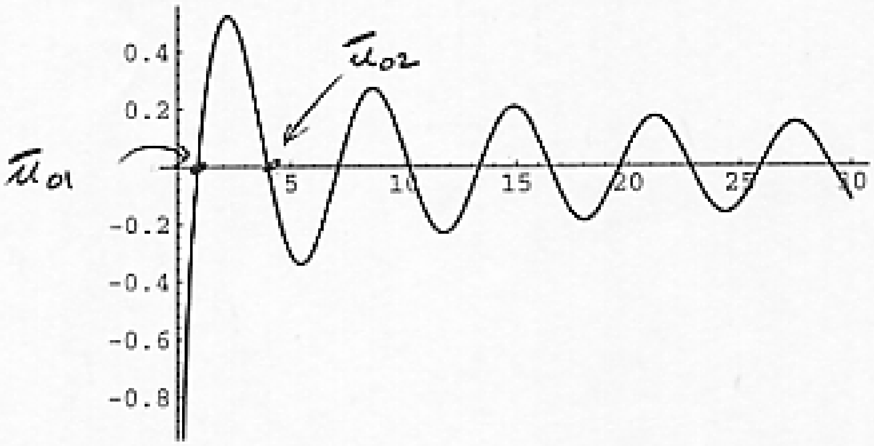
For $n > 0$, all the $J_n(u)$ vanish at the origin — but the origin is not counted as one of the zeroes u_{ns}.

RESERVE

In[13]:=

```
Plot[ BesselY[0,u],{u,0.1,30}]
```

$N_0(u)$ vs. u



Out[13]=

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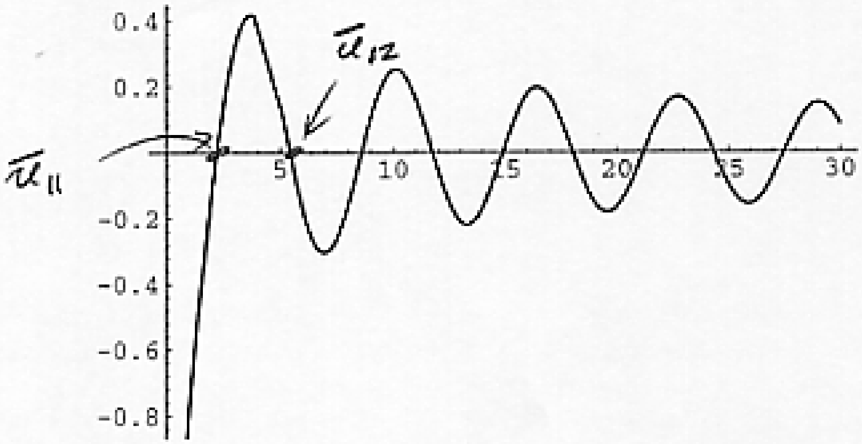
In[14]:=

```
Plot[ BesselY[1,u],{u,0.1,30}]
```

$N_1(u)$ vs. u



Out[14]=

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RESERVE

In[20]:=

```
tmp[u_]= (D[BesselJ[0,u],u])
```

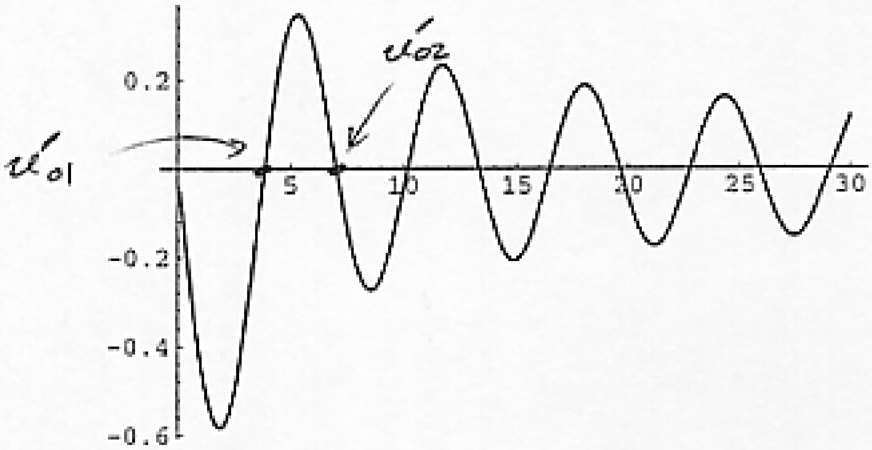
Out[20]=

```
BesselJ[-1, u] - BesselJ[1, u]
------------------------------
              2
```

In[21]:=

```
Plot[ tmp[u],{u,0.1,30}]
```

$J_0'(u)$ vs. u



Out[21]=

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```

In[22]:=

```
tmq[u_]= (D[BesselJ[1,u],u])
```

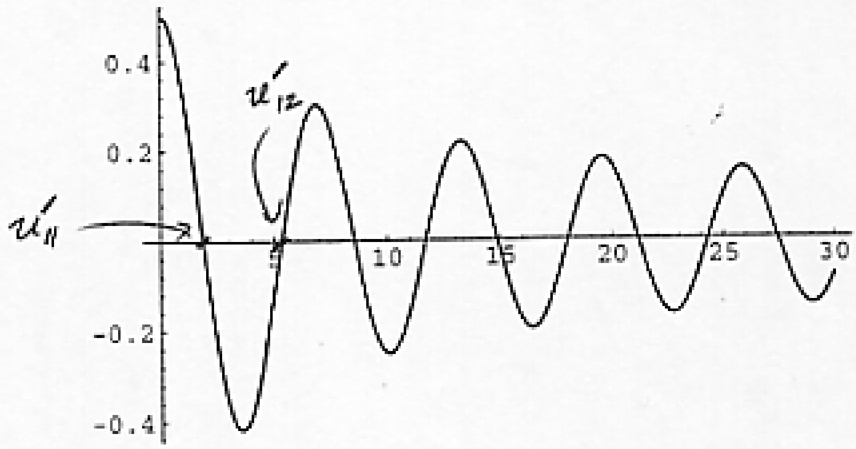
Out[22]=

```
BesselJ[0, u] - BesselJ[2, u]
-----------------------------
              2
```

In[23]:=

```
Plot[ tmq[u],{u,0.1,30}]
```

$J_1'(u)$ vs. u



Out[23]=

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RESERVE

There are enough relations among the Bessel functions to fill several texts and many courses. We will only need the following orthogonality relation:

$$\int_0^a \rho \, d\rho \; J_n\!\left(u_{ns}\frac{\rho}{a}\right) J_n\!\left(u_{ns'}\frac{\rho}{a}\right) = \frac{a^2}{2}\left[J_{n+1}\!\left(u_{ns}\frac{\rho}{a}\right)\right]^2 \delta_{ss'}$$

don't forget the "ρ" in the measure!

This is used to extract the expansion coefficients by the cylindrical analogue of Fourier's trick for sines and cosines.

—— End Lecture #12 ——

RESERVE