17 October 95

c) Spherical Boundaries

The Laplacian is

$$\nabla^2 \Phi(r,\theta,\varphi) = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\Phi\right)} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\Phi\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Phi$$

the first term can also be written as $\quad \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)$

i) boundary conditions depend only on $r \Rightarrow$ the potential can depend only on r: $\Phi = \Phi(r)$

This case describes <u>uniform</u> spheres held at a <u>constant</u> potential

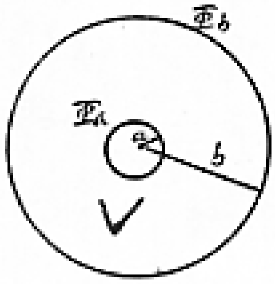
$$\nabla^2\Phi(r) = 0 = \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\Phi(r)\right] = 0$$

$$\Rightarrow r^2\frac{d}{dr}\Phi(r) = \text{constant} = -C_1$$

$$\Phi(r) = \frac{C_1}{r} + C_2$$

If the origin $(r=0)$ is included in V, then $C_1 = 0$.

Suppose V is the region between two spheres of radii $a<b$,

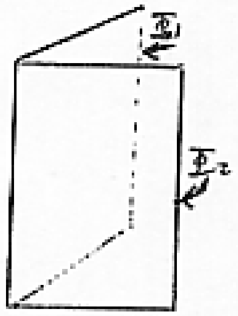


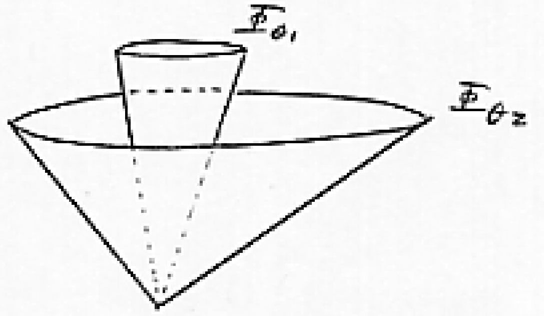
$$\left.\begin{aligned}\Phi(a) &= \frac{C_1}{a} + C_2 = \Phi_a\\ \Phi(b) &= \frac{C_1}{b} + C_2 = \Phi_b\end{aligned}\right\} \Rightarrow \text{determine } C_1 \text{ and } C_2$$

in V: $\Phi(r) = \frac{C_1}{r} + C_2$

RESERVE

We have already considered the case in which the boundary conditions depend on φ only. The geometry describes infinite wedges, which we considered in cylindrical coordinates.





We will not treat in detail the case in which the boundary conditions depend only on θ. The geometry describes infinite <u>cones</u>, each held at a constant potential.

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ii) Problems with <u>axial</u> symmetry $\Rightarrow$ no φ dependence

$$\Phi(r,\theta) = \frac{U(r)}{r}\,T(\theta)$$

the r function is written with an explicit factor of $\frac{1}{r}$ for later convenience

$$\nabla^2 \Phi(r,\theta) = 0$$

$$\frac{r^3 \nabla^2 \Phi(r,\theta)}{\Phi(r,\theta)} = 0 = \underbrace{r^2 \frac{U''(r)}{U(r)}}_{\text{function of } r \text{ alone}} + \underbrace{\frac{1}{T(\theta)\sin\theta}\left[\sin\theta\; T'(\theta)\right]'}_{\text{function of } \theta \text{ alone}}$$

RESERVE

In anticipation of later results, we will call the separation constant $\ell(\ell+1)$.

The separated differential equations are:

$$U''(r) - \frac{\ell(\ell+1)}{r^2} U(r) = 0 \Rightarrow U(r) = A r^{\ell+1} + B r^{-\ell}$$

$$\frac{d}{d\theta}\left[\sin\theta \frac{d}{d\theta} T(\theta)\right] + \ell(\ell+1) \sin\theta \, T(\theta) = 0$$

It is convenient to change variables

$$x \equiv \cos\theta \qquad T(\theta) \equiv P(\cos\theta)$$

$$dx = d(\cos\theta) = -\sin\theta \, d\theta$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P(x)\right] + \ell(\ell+1) P(x) = 0$$

This is the Legendre Differential Eq.

There are two types of solutions:

1) Legendre functions (of the first kind): $P_\ell(\cos\theta)$

these are oscillatory, like sines and cosines or like the Bessel functions J_n and N_n.

2) Legendre functions of the second kind: $Q_\ell(\cos\theta)$

these are like exponentials (sinh's and cosh's) or like the modified Bessel functions I_n and K_n.

The $Q_\ell(\cos\theta)$ functions <u>diverge</u> at $\theta = 0, \pi$

so if the region of interest is the <u>full</u> sphere including the polar axis, we must <u>exclude</u> the $Q_\ell(\cos\theta)$ solutions.

The Taylor series expansion of $P_\ell(\cos\theta)$ terminates, that is it has a finite number of terms, so the Legendre functions (of the first kind) are actually Legendre polynomials.

the first few are;

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$

These are normalized such that $P_\ell(1) = 1$.

The functions are orthogonal and complete on the interval $-1 \le x \le 1$ or equivalently $0 \le \theta \le \pi$.

Orthogonality

$$\int_{-1}^{+1} dx\, P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} = \int_0^{\pi} \sin\theta\, d\theta\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)$$

Completeness

Any "nice" function $f(x)$ can be expanded as

$$f(x) = \sum_{\ell=0}^{\infty} C_\ell\, P_\ell(x) \quad \text{on the interval } -1 \le x \le +1$$

One of the many identities involving Legendre polynomials is:

$$\frac{d}{dx} P_{\ell+1}(x) - \frac{d}{dx} P_{\ell-1}(x) = (2\ell+1) P_\ell(x)$$

So for a full sphere with azimuthal symmetry

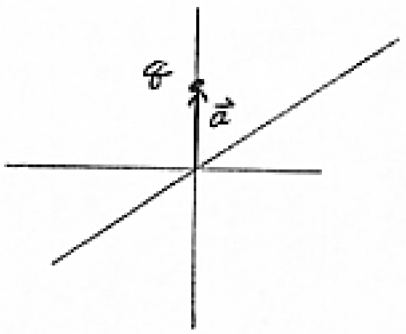
$$\Phi(r,\theta) = \frac{U(r)}{r} T(\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

If $r=0$ is included $\longrightarrow B_\ell = 0$

If $r=\infty$ is included $\Rightarrow A_\ell = 0$

Motivation

We have actually met the Legendre polynomials before. Consider the potential of a point charge q at position $\vec{a}$ from the origin. Choose the z-axis along $\vec{a}$.



$$\Phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{a}|} = \frac{q}{\sqrt{(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{a})}}$$

$$= \frac{q}{\sqrt{r^2 - 2ra\cos\theta + a^2}}$$

Now perform a multipole expansion of this potential. In order for the series to converge, we distinguish two regions.

RESERVE

1) $r < a$

$$\Phi(\vec{r}) = \frac{q}{a}\frac{1}{\sqrt{1-2\left(\frac{r}{a}\right)\cos\theta + \frac{r^2}{a^2}}}$$

Use the binomial expansion with expansion parameter $\frac{r}{a} < 1$

$$= \frac{q}{a}\left[1 + \left(\frac{r}{a}\right)\cos\theta + \left(\frac{r}{a}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right]$$

these are the Legendre Polynomials!

$$= \frac{q}{a}\sum_{\ell=0}^{\infty}\left(\frac{r}{a}\right)^{\ell} P_{\ell}(\cos\theta)$$

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2) $r > a$

$$\Phi(\vec{r}) = \frac{q}{r}\frac{1}{\sqrt{1-2\left(\frac{a}{r}\right)\cos\theta + \frac{a^2}{r^2}}}$$

Expansion parameter $\frac{a}{r} < 1$

$$= \frac{q}{r}\left[1 + \left(\frac{a}{r}\right)\cos\theta + \left(\frac{a}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right]$$

$$= \frac{q}{r}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^{\ell} P_{\ell}(\cos\theta)$$

We can describe an algorithm for isolating the ℓ^{th} Legendre Polynomial from the series above; Differentiate ℓ times with respect to the expansion parameter, divide by $\ell!$, and finally set the expansion parameter to zero.

Symbolically:

Expansion parameter $= t$

$$P_\ell(\cos\theta) = \frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial t^\ell} G(t,\cos\theta)\right]_{t=0}$$

where

$$G(t,\cos\theta) = \frac{1}{\sqrt{1-2t\cos\theta+t^2}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(\cos\theta)$$

is the Legendre polynomial generating function.

Now, a remarkable statement!

If we know the potential on the polar axis $\theta=0$, we can determine the potential for all space. This is an incredible savings in time and effort!

Suppose that, by any means whatsoever, you determine the potential along a line to be $f(r)$, where r is the distance from some origin. Then choose the polar axis to coincide with the line, and

$$\Phi(r,0) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] = f(r)$$

known function

we used: $P_\ell(\cos 0) = P_\ell(1) \equiv 1$

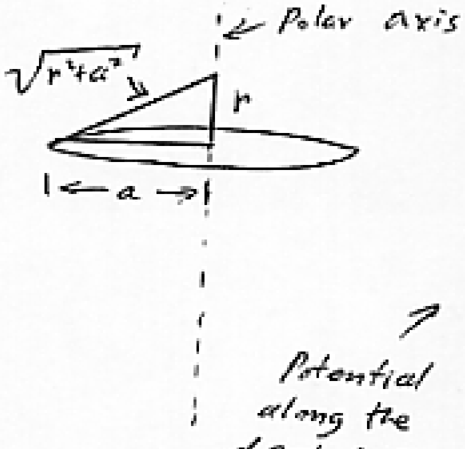
The sum over ℓ is called a "Laurent Expansion" of the function $f(r)$.

A Laurent Expansion is simply a Taylor series with both <u>positive</u> and <u>negative</u> powers of r.

We can read off the coefficients A_ℓ and B_ℓ from the Laurent expansion of $f(r)$. Then it is trivial to "take the solution off axis." We simply put the $P_\ell(\cos\theta)$ back in the sum!

An example will illustrate the power and beauty of this technique!

Suppose we have a ring of charge <u>q</u> and radius <u>a</u>.



For a point a distance <u>r</u> from the origin (at the center of the ring), we have

$$\Phi(r) = \frac{q}{\sqrt{r^2+a^2}} = f(r)$$

Potential along the dashed line

$$= \begin{cases} \dfrac{q}{a}\dfrac{1}{\sqrt{1+r^2/a^2}}, & r<a \\[2ex] \dfrac{q}{r}\dfrac{1}{\sqrt{1+a^2/r^2}}, & r>a \end{cases}$$

The Laurent expansion of

$$\frac{1}{\sqrt{1+x^2}} \text{ is } \sum_{n=0}^{\infty} (-1)^n C_n x^{2n}$$

(convergent if $x^2<1$.)

This is in fact a Taylor expansion because only positive powers are needed.

$$C_0 = 1 , \quad C_n = \frac{(2n-1)!!}{2^n n!} \quad n \geq 1$$

where $m!! \equiv m(m-2)(m-4)\cdots 1$

Now we can go <u>off axis</u>:

$$\Phi(r,\theta) = \begin{cases} \frac{q}{a} \sum_{n=0}^{\infty} (-1)^n C_n \left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta) , & r<a \\ \frac{q}{r} \sum_{n=0}^{\infty} (-1)^n C_n \left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta) , & r>a \end{cases}$$

—— End Lecture # 14 ——