iii) Boundary conditions depend on all 3 coordinates

$$\Phi(r,\theta,\varphi) = \frac{U(r)}{r} T(\theta) F(\varphi) = R(r) T(\theta) F(\varphi)$$

$$\frac{r^3 \sin^2\theta \, \nabla^2 \Phi(r,\theta,\varphi)}{\Phi(r,\theta,\varphi)} = 0 = \underbrace{r^2 \sin^2\theta \left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2 \sin\theta \, T}\frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right)\right]}_{\text{function of } r \text{ and } \theta} + \underbrace{\frac{F''}{F}}_{\text{Function of } \varphi}$$

First separation constant : m^2

$$F''(\varphi) + m^2 F(\varphi) = 0 \quad \Rightarrow \quad F(\varphi) = e^{\pm i m \varphi}$$

If the physical Volume includes the full range of φ, $0 \to 2\pi$ then $\underline{m}$ is an integer $\ldots -2, -1, 0, 1, 2, \ldots$

This guarantees periodicity in φ with period 2π.

$$r^2 \sin^2\theta \left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2 \sin\theta \, T}\frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right)\right] = m^2$$

This equation can be separated. Call the second separation constant $\ell(\ell+1)$.

$$U''(r) - \frac{\ell(\ell+1)}{r^2} U(r) = 0 \quad \Rightarrow \quad U(r) = A r^{\ell+1} + B r^{-\ell}$$

$$R(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}$$

The θ equation is

$$\frac{1}{\sin\theta}(\sin\theta \, T')' + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] T = 0$$

The θ equation is more manageable with a change of variable

$x = \cos\theta \qquad dx = -\sin\theta \, d\theta \qquad T(\theta) = P(\cos\theta) = P(x)$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P(x) = 0$$

this is the <u>generalized</u> Legendre equation.

The solutions are <u>associated</u> Legendre functions

First kind: $P_\ell^m(\cos\theta)$ - oscillatory, like sin, cos or J_n, N_n

ℓ is a positive integer and $-\ell \le m \le +\ell$.

Second kind: $Q_\ell^m(\cos\theta)$ - growth and decay, like exponentials, I_n, K_n.

these diverge at $\theta = 0, \pi$ - the north and south poles, so if the polar axis is in V, the Q_ℓ^m functions must be excluded.

Before we put all the pieces of the potential together, let us consider solutions independent of r.

$\Phi \propto T(\theta)F(\varphi)$ with a normalization, these are the <u>spherical Harmonics</u>

$$Y_{\ell m}(\theta,\varphi) = \underbrace{\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}}_{\text{normalization}} \underbrace{P_\ell^m(\cos\theta)}_{T(\theta)} \underbrace{e^{im\varphi}}_{F(\varphi)}$$

The Spherical Harmonics satisfy several relations:

orthonormality

$$\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta}_{\int d^2\Omega}\; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

Closure

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta^2(\Omega-\Omega')$$

$$= \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

$$= \delta(\varphi-\varphi')\,\frac{\delta(\theta-\theta')}{\sin\theta}$$

Completeness for any function $g(\theta,\varphi)$:

$$g(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} A_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

where $A_{\ell m} = \int d^2\Omega\; Y^*_{\ell m}(\theta,\varphi)\, g(\theta,\varphi)$

The first few Spherical Harmonics are:

$Y_{00} = \frac{1}{\sqrt{4\pi}}$ $\quad \ell=0, m=0$

$Y_{11} = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\varphi}$ $\quad \ell=1, m=+1$

$Y_{10} = \sqrt{\frac{3}{4\pi}}\,\cos\theta$ $\quad \ell=1, m=0$

$Y_{1,-1} = +\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\varphi}$ $\quad \ell=1, m=-1$

$$Y_{\ell,-m} = (-1)^m\, Y^*_{\ell m}$$

RESERVE

The general solution for all three coordinates is:

$$\Phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left[A_{\ell m}\, r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right] Y_{\ell m}(\theta,\varphi)$$

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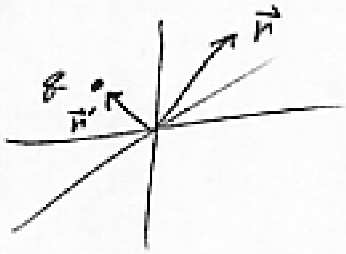
Motivation

Put a point charge at the source point $\vec{r}'$, not on the polar axis. The potential at the field point $\vec{r}$ is:



$$\Phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} = \frac{q}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2}}$$

Call the angle between $\vec{r}$ and $\vec{r}'$; γ

$$\Phi(\vec{r}) = \frac{q}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}}$$

$$= \begin{cases} \dfrac{q}{r'} \displaystyle\sum_{\ell=0}^{\infty} \left(\frac{r}{r'}\right)^{\ell} P_\ell(\cos\gamma) & r < r' \\ \dfrac{q}{r} \displaystyle\sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\gamma) & r' < r \end{cases}$$

Now we would like to express $(\cos\gamma)$ as a function of θ, θ', φ, and φ':

$$\cos\gamma = \frac{\vec{r}\cdot\vec{r}'}{rr'} = \frac{xx' + yy' + zz'}{rr'}$$

$$= \frac{rr'\left[\sin\theta\sin\theta'(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') + \cos\theta\cos\theta'\right]}{rr'}$$

or, using a trigonometric identity:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

This relation is the famous trigonometric addition formula. It can be used to prove the addition formula for spherical harmonics:

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$$

There is also a sum rule for spherical harmonics:

$$\sum_{m=-\ell}^{+\ell} |Y_{\ell m}(\theta,\varphi)|^2 = \frac{2\ell+1}{4\pi} = \sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta,\varphi)\,Y_{\ell m}(\theta,\varphi)$$

//

We now derive the Dirichlet Green functions for spherical geometries:

1. interior of a sphere of radius a: $r, r' < a$
2. exterior of a sphere of radius a: $r, r' > a$
3. the region between two spheres of radii $a<b$: $a < r, r' < b$

We solved cases 1 and 2 previously during our study of the method of images:

$$G_1(\vec r,\vec r') = \frac{1}{\sqrt{r^2 - 2\vec r\cdot\vec r' + r'^2}} - \frac{a}{\sqrt{r^2 r'^2 - 2a^2\vec r\cdot\vec r' + a^4}}$$

Let γ be the angle between $\vec{r}$ and $\vec{r}'$.

$$G_D(\vec{r},\vec{r}') = \frac{1}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}} - \frac{a}{\sqrt{r^2r'^2 - 2a^2rr'\cos\gamma + a^4}}$$

① Interior: $\frac{rr'}{a^2} \le 1$ this is the expansion parameter.

The first term is

$$\frac{1}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}} = \begin{cases} \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\gamma) & , r < r' \\ \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos\gamma) & , r' < r \end{cases}$$

these two cases can be summarized by

$$\frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma)$$

where $r_> \equiv \max[r, r']$ and $r_< \equiv \min[r, r']$

the second term in $G_D(\vec{r},\vec{r}')$ is:

$$-\frac{1}{a} \sum_{l=0}^{\infty} \left(\frac{r_< r_>}{a^2}\right)^l P_l(\cos\gamma)$$

for both $r<r'$ and $r>r'$ since $rr' = r_< r_>$ is symmetric in $r_<, r_>$

① $$G_D(\vec{r},\vec{r}')_{\text{interior}} = \sum_{l=0}^{\infty} \left\{ \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l - \frac{1}{a}\left(\frac{r_< r_>}{a^2}\right)^l \right\} P_l(\cos\gamma)$$

Now we use the addition formula to express $P_l(\cos\gamma)$ in terms of θ, θ', φ, and φ'

RESERVE

① $$\underset{\text{interior}}{G_D(\vec{r},\vec{r}\,')} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\left\{\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell} - \frac{1}{a}\left(\frac{r_< r_>}{a^2}\right)^{\ell}\right\} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

---

② For the exterior problem, the first term remains unchanged while the expansion parameter in the second term becomes:

$$\frac{a^2}{r r'} \leq 1 \qquad \text{since both } r, r' \geq a$$

② $$\underset{\text{exterior}}{G_D(\vec{r},\vec{r}\,')} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\left\{\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell} - \frac{1}{a}\left(\frac{a^2}{r_< r_>}\right)^{\ell+1}\right\} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

---

③ This result is new:

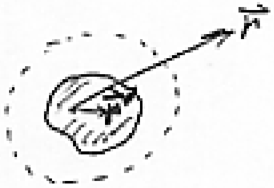
$$\underset{\substack{\text{between}\\ a<b}}{G_D(\vec{r},\vec{r}\,')} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\, \frac{\left(r_<^{\ell} - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)}{1-\left(\frac{a}{b}\right)^{2\ell+1}}\, Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

It is easy to verify that

$$\underset{\substack{\text{between}\\ a<b}}{G_D(\vec{r},\vec{r}\,')} = 0 \quad \text{for} \quad r \ \underline{\text{or}}\ r' = a \ \underline{\text{or}}\ b$$

Spherical Multipole Moment Tensors

Consider a region of charge bounded by a sphere. Outside the sphere, the potential can be expanded as a convergent series in the parameter $\frac{r'}{r} < 1$. Here, $\vec{r}\,'$ is the source point and $\vec{r}$ is the field point.



$$\Phi(\vec{r}) = \int_V dV' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= \int_V dV' \frac{\rho(\vec{r}\,')}{\sqrt{r^2 + 2rr'\cos\gamma + r'^2}}$$

$$= \int_V dV' \rho(\vec{r}\,') \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos\gamma)$$

$$= \int_V dV' \rho(\vec{r}\,') \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)$$

$$\equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} Q_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta,\varphi)$$

The moments Q_{lm} contain all the information about the charge — all the primed variables.

$$Q_{lm} \equiv \int dV' \rho(\vec{r}\,') Y^*_{lm}(\theta',\varphi') \, r'^l \qquad \left(\begin{array}{c}\text{complex,} \\ \text{in general}\end{array}\right)$$

There are $2l+1$ components for a given l — the same number of independent elements in the irreducible Cartesian multipole moments. The spherical multipole moments are automatically irreducible!

— end 15 —