Electrostatic Energy in a Dielectric Medium

A long time ago, we derived the energy in vacuum

$$U_{vacuum} = \frac{1}{8\pi}\int dV\, E^2(\vec{r}) = \frac{1}{2}\int dV\, \rho(\vec{r})\, \Phi(\vec{r})$$

avoid infinite self-energies

We now wish to derive the analogous quantity in a dielectric. We begin with some pre-existing distribution of charge which gives rise to a potential $\Phi(\vec{r})$. Next, we bring an infinitesimal piece of true charge $\delta\rho$ in from $r=$ infinity and deposit it at position $\vec{r}$. The infinitesimal change in the energy of the system is the negative of the work done:

$$\delta U = -\delta W = \int dV\, \delta\rho(\vec{r})\, \Phi(\vec{r})$$

We do not include in $\Phi(\vec{r})$ the effects of changing the pre-existing charge distribution since this would result in a second-order change in U.

Substitute: $\delta\rho(\vec{r}) = \frac{1}{4\pi}\,\delta(\vec{\nabla}\cdot\vec{D}(\vec{r})) = \frac{1}{4\pi}\,\vec{\nabla}\cdot\delta\vec{D}(\vec{r})$

$$\delta U = \frac{1}{4\pi}\int dV\, \left[\vec{\nabla}\cdot\delta\vec{D}(\vec{r})\right]\Phi(\vec{r})$$

Integrate this expression by parts:

$$\delta U = \frac{1}{4\pi}\int dV \left[\vec{\nabla}\cdot(\delta\vec{D}\,\Phi) - \delta\vec{D}\cdot\vec{\nabla}\Phi \right]$$

$$= \frac{1}{4\pi}\cancelto{0}{\oint dS\,\hat{n}\cdot\delta\vec{D}\,\Phi} - \frac{1}{4\pi}\int dV\,\delta\vec{D}\cdot\vec{\nabla}\Phi$$

The volume of integration is all space, so the surface in the first integral must be the surface at infinity. We assume that the charge distribution does not extend this far and that the potential vanishes in this limit.

In the second term, we use $\vec{E} = -\vec{\nabla}\Phi$

$$\delta U = \frac{1}{4\pi}\int dV\,\delta\vec{D}\cdot\vec{E}$$

Now we can build up a distribution of <u>true</u> charge by adding up the infinitesimal pieces:

$$U(\vec{D}) - U(0) = \frac{1}{4\pi}\int dV \int_0^{\vec{D}} d\vec{D}\cdot\vec{E}(\vec{D})$$

where we integrated <u>formally</u> from $\vec{D}=0$ to some finite displacement field $\vec{D}$. The macroscopic electric field depends on $\vec{D}$.

We exclude the electrostatic energy involved in assembling the dielectric (the <u>bound</u> charge). In other words, we choose $U=0$ when $\vec{D}=0 \Rightarrow$ when there is no <u>true</u> charge, there is no energy.

For a general dielectric medium, this is the final expression:

$$U(\vec{D}) = \frac{1}{4\pi}\int dV \int_0^{\vec{D}} d\vec{D} \cdot \vec{E}(\vec{D})$$

but for a linear dielectric, we can carry the analysis farther. We will allow for anisotropic media:

$$D_i = \sum_j \epsilon_{ij} E_j$$

where ϵ_{ij} is the symmetric dielectric tensor $\epsilon_{ij} = \epsilon_{ji}$. Then the integrand is:

$$d\vec{D} \cdot \vec{E} = \sum_i dD_i E_i = \sum_i \sum_j d(\epsilon_{ij} E_j) E_i$$

$$= \sum_i \sum_j \epsilon_{ij} E_i \, dE_j$$

Use the symmetry of ϵ_{ij} to write

$$= \frac{1}{2} \sum_i \sum_j d(E_i \epsilon_{ij} E_j)$$

$$= \frac{1}{2} \sum_i d(E_i D_i) = \frac{1}{2} d(\vec{E} \cdot \vec{D})$$

So for a linear medium we have

$$U = \frac{1}{8\pi} \int dV \, \vec{E}(\vec{r}) \cdot \vec{D}(\vec{r})$$

this is the analogous expression that we sought

If we substitute $\vec{E} = -\vec{\nabla}\Phi$ and $\vec{\nabla} \cdot \vec{D} = 4\pi \rho_{true}$ we get

$$U = \frac{1}{2} \int dV \, \rho_{true}(\vec{r}) \, \Phi(\vec{r})$$

in this form, be careful to avoid infinite self-energies.

To see why the dielectric tensor is symmetric, look at the energy density $u(\vec{r})$:

$$U = \int dV \, u(\vec{r})$$

$$u(\vec{D}) = \frac{1}{4\pi} \int_0^{\vec{D}} d\vec{D} \cdot \vec{E}(\vec{D})$$

This is a line integral in $\vec{D}$-space. If we assume that the energy density is independent of the path and depends only on the end points (there is no hysteresis), then the differential form of the equation above is:

$$\vec{\nabla}_{\vec{D}} \, u = \frac{\vec{E}(\vec{D})}{4\pi} \quad \text{or} \quad \frac{E_i}{4\pi} = \frac{\partial U}{\partial D_i}$$

Make a Taylor expansion of E_i ~~as~~ a function of $\vec{D}$:

$$E_i(\vec{D}) = \cancelto{0}{E_i(0)} + \sum_j \left.\frac{\partial E_i}{\partial D_j}\right|_{\vec{D}=0} D_j + \cdots$$

To avoid hysteresis, there can be no ferro-electrics in the problem, so $E_i(0) = 0$.

$$E_i = \sum_j \frac{\partial E_i}{\partial D_j} D_j$$ substitute our expression for E_i above

$$E_i = \sum_j \frac{\partial^2 U}{\partial D_i \partial D_j} (4\pi) \, D_j \equiv \sum_j \lambda_{ij} D_j$$

A model for quark confinement

Suppose we had a fluid dielectric medium of dielectric constant ϵ and fluid pressure B.

The energy required to make a spherical hole (bubble) of radius R is

$$U_{hole} = \int_0^R 4\pi r^2 \, dr \, B = \frac{4\pi}{3} R^3 B$$

B is the "energy density" of the fluid.

The hole has the dielectric constant of the vacuum $\epsilon_0 = 1$. Put a point charge $\underline{q}$ at the center of the bubble. This is the only true charge in the problem. Then

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho = 4\pi q \, \delta^3(\vec{r}) \Rightarrow \oint ds \, \hat{n} \cdot \vec{D} = 4\pi q$$

and by spherical symmetry

$$\vec{D}_{(\vec{r})} = \frac{q}{r^2} \hat{r}$$

So the electric field is:

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{r^2} \hat{r} & , \ r < R \quad (\text{in vacuum, } \vec{E} = \vec{D}) \\ \dfrac{q}{\epsilon r^2} \hat{r} & , \ r > R \end{cases}$$

The electrostatic energy of the system is:

$$U_{el} = \frac{1}{8\pi}\int dV\, \vec{E}\cdot\vec{D}$$ but this includes the infinite self-energy of the point charge.

If we remove the infinity, the energy is:

$$U_{el} = \frac{1}{8\pi}\int dV\, \vec{E}\cdot\vec{D} - \frac{1}{8\pi}\int dV\, \vec{D}^2 = \frac{1}{8\pi}\int_{\substack{\text{All}\\ \text{Space}}} dV\, (\vec{E}-\vec{D})\cdot\vec{D}$$

$$= \frac{1}{8\pi}\int_{\substack{\text{All}\\ \text{Space}}} dV\, D^2\left(\frac{1}{\epsilon(\vec{r})}-1\right) \quad \text{where} \quad \epsilon(\vec{r}) = \begin{cases} 1 & , r<R \\ \epsilon & , r>R \end{cases}$$

$$= \frac{1}{8\pi}\int_{r>R} 4\pi r^2 dr \left(\frac{1}{\epsilon}-1\right)\frac{q^2}{r^4} \quad \text{since } \left(\frac{1}{\epsilon(\vec{r})}-1\right) = 0 \text{ for } r<R,$$

$$= \frac{1}{2}\left(\frac{1}{\epsilon}-1\right)\frac{1}{R}$$

The total energy of the system is:

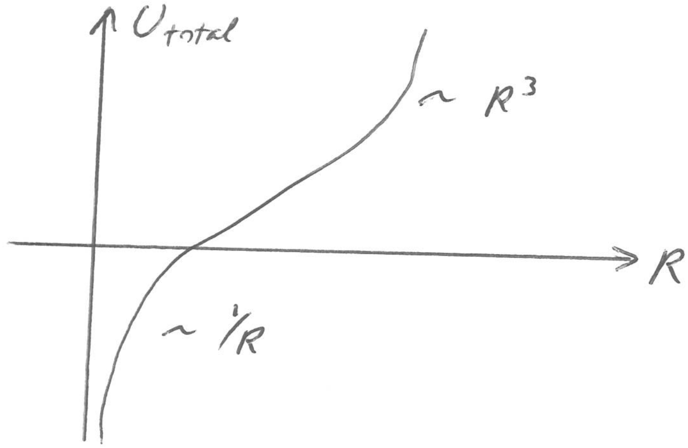
$$U_{Total} = U_{hole} + U_{el} = \frac{4\pi}{3}R^3 B + \frac{1}{2}\left(\frac{1}{\epsilon}-1\right)\frac{1}{R}$$

Now we can ask: Will a hole form spontaneously?

Some graphs will show the answer:

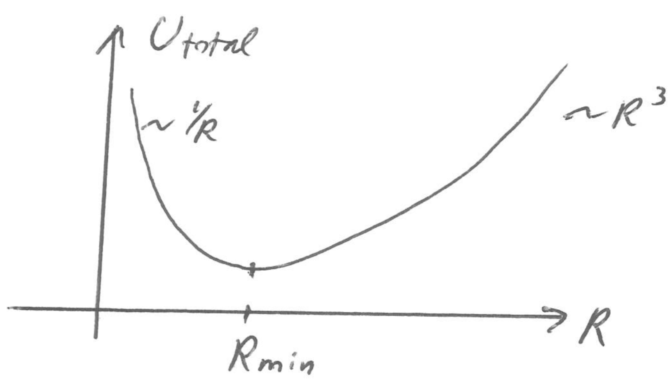
For normal, that is "para", dielectrics $\epsilon > 1$

And a graph of U_{total} vs. Hole radius R looks like:



We have no stable minimum of energy at non-zero R. Any hole made will collapse to zero radius.

But suppose the medium were a dia-electric, that is $0 < \epsilon < 1$. Then the graph is



R_{min} is the radius of the bubble that minimizes the energy

$$\left.\frac{\partial U}{\partial R}\right|_{R_{min}} = 0$$

$$R_{min} = \left[\frac{\left(\frac{1}{\epsilon} - 1\right) q^2}{8\pi B} \right]^{1/4}$$

and

$$U(R_{min}) = \frac{1}{3}\left(\frac{1}{\epsilon} - 1\right)^{3/4} \left(\frac{4\pi B}{q^2}\right)^{1/4}$$

So a spherical hole will spontaneously form around a point charge in a dielectric medium.

Now suppose that the medium is perfectly dia-electric, that is $\epsilon = 0$. In this case, the fluid is a magnetic superconductor in analogy with an electric superconductor (like tin) in which the magnetic permeability μ vanishes.

For vanishing dielectric constant $\epsilon = 0$, the energy of the system is infinite and the hole would have infinite radius:

$$U_{min} \xrightarrow[\epsilon \to 0]{} \infty$$

$$R_{min} \xrightarrow[\epsilon \to 0]{} \infty$$

To avoid this catastrophe, the system must be neutral, $q = 0$. The displacement $\vec{D}$ will be excluded from the diaElectric medium — this is the analog of the Meissner effect.

This electrostatics problem serves as a model for quark confinement. The role of electric charge is taken over by a property called color and the electric field is replaced by the chromoelectric field. Charge neutrality becomes a colorless state or "color singlet." The dielectric fluid is the QCD vacuum. But we have just traded one supposition for another. Is the QCD vacuum a perfect dia-electric?

II. Magnetostatics

A) Basic Aspects of Magnetostatics

The basic magnetic force between two moving electric charges as inferred from experiment is:

$$\vec{F}_{1\,\text{on}\,2} = q_1 q_2 \frac{\vec{v}_2}{c} \times \left[\frac{\vec{v}_1}{c} \times \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \right]$$

This is the analogue of Coulomb's law for <u>electro</u>-statics.

q_1 and q_2 are the electric charges, $\vec{r}_1$ and $\vec{r}_2$ are their position vectors, and $\vec{v}_1$ and $\vec{v}_2$ are their velocities $\vec{v}_i = \frac{d\vec{r}_i}{dt}$.

Notice that this force law violates Newton's third law: $\vec{F}_{1\,\text{on}\,2} \neq -\vec{F}_{2\,\text{on}\,1}$!

We define the magnetic field $\vec{B}$ by

$$\vec{F}_{1\,\text{on}\,2} = q_2 \frac{\vec{v}_2}{c} \times \vec{B}_1(\vec{r}_2)$$ where

$$\vec{B}_1(\vec{r}_2) = q_1 \frac{\vec{v}_1}{c} \times \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3}$$ source point

field point

$\vec{B}_1(\vec{r}_2)$ is the magnetic field <u>at</u> $\vec{r}_2$ due to the moving charge q_1 <u>at</u> $\vec{r}_1$.

The vector magnetic force obeys the superposition principle:

$\vec{r}$ = field point
$\vec{r}_i$ = source points

$$\vec{B}(\vec{r}) = \sum_{i=1}^{N} q_i \frac{\vec{v}_i}{c} \times \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

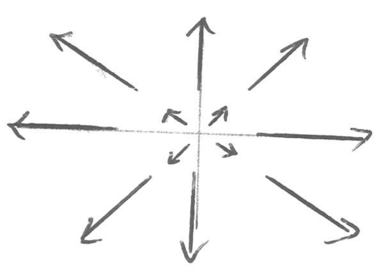
$$= -\sum_{i=1}^{N} q_i \frac{\vec{v}_i}{c} \times \vec{\nabla} \frac{1}{|\vec{r}-\vec{r}_i|}$$

We now use a vector calculus identity for a scalar field $\psi(\vec{r})$ and a vector field $\vec{w}(\vec{r})$;

$$\vec{\nabla} \times \left[\psi(\vec{r})\, \vec{w}(\vec{r})\right] = \psi(\vec{r})\, \vec{\nabla} \times \vec{w}(\vec{r}) - \vec{w}(\vec{r}) \times \vec{\nabla} \psi(\vec{r})$$

let $\psi = \dfrac{1}{|\vec{r}-\vec{r}_i|}$ and $\vec{w} = \vec{v}$

and notice that $\vec{\nabla} \times \vec{r} = 0$. You can prove this easily in Cartesian coordinates, or graph the vector function $\vec{r}$:



and <u>see</u> that this field has <u>no</u> <u>curl</u>.

then since $\vec{v} = \dfrac{d\vec{r}}{dt}$, we have

$$\vec{\nabla} \times \vec{v} = 0$$

$$\vec{B}(\vec{r}) = \sum_{i=1}^{N} q_i \frac{1}{c} \vec{\nabla} \times \left(\frac{\vec{v}_i}{|\vec{r} - \vec{r}_i|} \right)$$

$$= \vec{\nabla} \times \vec{A}(\vec{r})$$

where

$$\vec{A}(\vec{r}) = \frac{1}{c} \sum_{i=1}^{N} q_i \frac{\vec{v}_i}{|\vec{r} - \vec{r}_i|}$$

is the <u>vector Potential</u>

End Lecture #18