9 November 95

Magnetic Multipole Expansion

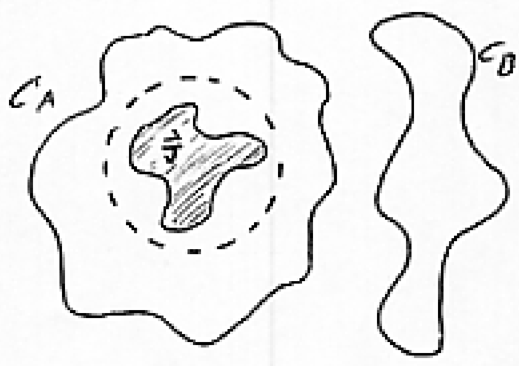
The problem of finding $\vec{B}$ in a region away from currents suggests that we make a multipole expansion similar to what was done in electrostatics. Because $\vec{B}$ is determined from a <u>vector</u> potential $\vec{A}$, however, the multipole expansion could be considerably more involved. We will proceed in a different way which will make the whole analysis completely analogous to the electrostatic case.

Suppose the current density $\vec{J}(\vec{r}')$ is confined to a sphere of radius <u>a</u>. Then outside the sphere:

$$\vec{\nabla}\times\vec{B}(\vec{r}) = 0 \qquad \left(\text{since } \tfrac{4\pi}{c}\vec{J}(\vec{r}) = 0 \text{ there}\right)$$
$$r > a$$

By Stoke's Theorem, we have

$$\int_S dS\, \hat{n}\cdot\vec{\nabla}\times\vec{B}(\vec{r}) = \oint_C d\vec{r}_1\cdot\vec{B}(\vec{r}_1) = 0$$

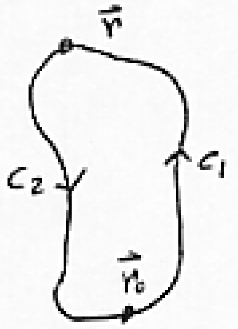


where C is any <u>closed</u> curve which does <u>not</u> intersect the mathematical sphere and S is any <u>open</u> surface bounded by C no points of which lie within the sphere.

$\vec{r}'$ = source point
$\vec{r}$ = field point

$\vec{r}_1$ = dummy integration variable

We break the contour into two pieces : C_1 and C_2 separated by two points $\vec{r}$ and $\vec{r}_0$.



$$0 = \oint_C d\vec{r}_1 \cdot \vec{B}(\vec{r}_1)$$

$$= \int_{\vec{r}_0\ C_1}^{\vec{r}} d\vec{r}_1 \cdot \vec{B}(\vec{r}_1) + \int_{\vec{r}\ C_2}^{\vec{r}_0} d\vec{r}_1 \cdot \vec{B}(\vec{r}_1)$$

therefore

$$\int_{\vec{r}_0\ C_1}^{\vec{r}} d\vec{r}_1 \cdot \vec{B}(\vec{r}_1) = \int_{\vec{r}_0\ C_2}^{\vec{r}} d\vec{r}_1 \cdot \vec{B}(\vec{r}_1)$$

and hence the integral is independent of the path ; it depends only on the endpoints.

We define :

$$\Phi_m(\vec{r}) - \Phi_m(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{r}_1 \cdot \vec{B}(\vec{r}_1)$$

$\Phi_m(\vec{r})$ is the magnetic scalar potential.

$$\vec{B}(\vec{r}) = -\vec{\nabla}\Phi_m(\vec{r})$$

This result is apparently contradictory since we also have $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$ everywhere which seems to imply that $\nabla^2 \Phi_m(\vec{r}) = 0$ everywhere and hence $\vec{B}(\vec{r}) = 0$ everywhere ! The way out of the dilemma is that

$$\vec{B}(\vec{r}) = -\vec{\nabla}\Phi_m(\vec{r}) \qquad \text{only } \underline{\text{outside}} \text{ the sphere,}$$

where the current density vanishes.

We now <u>choose</u> $\Phi_m(\infty) = 0$ and <u>choose</u> the path of integration over $\vec{r}_1$ to be along $\hat{r}$ out to infinity.

$$\Phi_m(\vec{r}) = -\int_{\infty}^{r} dr_1 \; \hat{r}\cdot\vec{B}(r_1\hat{r}) \qquad \text{where } \vec{r}_1 = r_1\hat{r}$$

This particular choice of path doesn't affect the result because the integral is path-independent.

Next we use the following expression for the magnetic field,

$$\vec{B}(\vec{r}_1) = \vec{\nabla}_1\times\vec{A}(\vec{r}_1) = -\frac{1}{c}\int dV'\,\vec{J}(\vec{r}\,')\times\vec{\nabla}_1\frac{1}{|\vec{r}_1-\vec{r}\,'|}$$

where $\vec{\nabla}_1$ means a derivative with respect to $\vec{r}_1$ coordinates.

$$= +\frac{1}{c}\int dV'\,\vec{J}(\vec{r}\,')\times\vec{\nabla}'\frac{1}{|\vec{r}_1-\vec{r}\,'|}$$

So far, we have

$$\Phi_m(\vec{r}) = -\int_{\infty}^{r} dr_1\,\hat{r}\cdot\int dV'\,\frac{\vec{J}(\vec{r}\,')}{c}\times\vec{\nabla}'\frac{1}{|r_1\hat{r}-\vec{r}\,'|}$$

Use the cyclicity of the triple product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$\Phi_m(\vec r) = -\int_\infty^r dr_1 \int dV' \frac{\vec J(\vec r')}{c} \cdot \left[\vec\nabla' \frac{1}{|r_1\hat r - \vec r'|} \times \hat r\right]$$

Concentrate on the term in square brackets

$$\vec\nabla' \frac{1}{|r_1\hat r - \vec r'|} \times \hat r = \frac{r_1\hat r - \vec r'}{|r_1\hat r - \vec r'|^3} \times \hat r = \frac{-\vec r' \times \hat r}{|r_1\hat r - \vec r'|^3}$$

where we used $\hat r \times \hat r = 0$. Now, since $\vec r' \times \vec r' = 0$ also, we can write

$$= \frac{1}{r_1} \frac{(r_1\hat r - \vec r')\times \vec r'}{|r_1\hat r - \vec r'|^3} = \frac{1}{r_1} \vec\nabla' \frac{1}{|r_1\hat r - \vec r'|} \times \vec r'$$

Thus

$$\Phi_m(\vec r) = -\frac{1}{c}\int_\infty^r \frac{dr_1}{r_1} \int dV' \vec J(\vec r') \cdot \left[\vec\nabla' \frac{1}{|r_1\hat r - \vec r'|} \times \vec r'\right]$$

use the cyclicity again

$$= -\frac{1}{c}\int_\infty^r \frac{dr_1}{r_1} \int dV' \left[\vec r' \times \vec J(\vec r')\right] \cdot \vec\nabla' \frac{1}{|r_1\hat r - \vec r'|}$$

We are trying to collect all the source coordinates $\vec r'$ together to facilitate the multipole expansion.

Next, integrate by parts and discard the surface term as usual by choosing the surface far away from the current distribution.

$$\Phi_m(\vec{r}) = \frac{1}{c}\int dV'\, \vec{\nabla}' \cdot \left[\vec{r}' \times \vec{J}(\vec{r}')\right] \int_{\infty}^{r} dr_1\, \frac{1}{r_1}\, \frac{1}{|r_1\hat{r} - \vec{r}'|}$$

Now since $\vec{J}(\vec{r}')$ vanishes for $r' > a$ and r is always greater than $\underline{a}$ (field point outside the sphere), and finally r_1 ranges from r to infinity, we have

$$r' < a < r \leq r_1$$

so we can expand $\frac{1}{r_1}\frac{1}{|r_1\hat{r} - \vec{r}'|}$ in the parameter $\left(\frac{r'}{r_1}\right) < 1$.

$$\frac{1}{r_1}\frac{1}{|r_1\hat{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{(r')^\ell}{r_1^{\ell+2}} P_\ell(\cos\gamma)$$

where γ is the angle between $\hat{r}$ and $\vec{r}'$. So

$$\int_{\infty}^{r} dr_1\, \frac{1}{r_1}\frac{1}{|r_1\hat{r} - \vec{r}'|} = -\sum_{\ell=0}^{\infty} \frac{(r')^\ell}{r^{\ell+1}} \frac{1}{\ell+1} P_\ell(\cos\gamma)$$

The last step is to use the addition formula for spherical harmonics

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

where θ and φ are the polar and azimuthal angles of $\vec{r}$ and θ' and φ' similarly for $\vec{r}'$.

The magnetic multipole expansion for the magnetic scalar potential is:

$$\Phi_m(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} \mathfrak{M}_{\ell m} \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi)$$

this is exactly the same form as the electrostatic multipole expansion.

The magnetic multipole moment is

$$\mathfrak{M}_{\ell m} = \frac{-1}{(\ell+1)c} \int dV' (r')^{\ell} Y^*_{\ell m}(\theta', \varphi') \vec{\nabla}' \cdot [\vec{r}' \times \vec{J}(\vec{r}')]$$

which is <u>not</u> of the same form as the electric multipole moment $Q_{\ell m}$.

Notice that $\mathfrak{M}_{\ell m}$ (and $Q_{\ell m}$) contain all the information about the <u>source</u> distribution — all the primed coordinates $\vec{r}'$.

The role of magnetic charge density is being played by

$$\vec{\nabla}' \cdot [\vec{r}' \times \vec{J}(\vec{r}')] \equiv \rho_m(\vec{r}')$$

We could also make a multipole expansion in <u>Cartesian</u> tensors:

(reducible)

$$\Phi_m(\vec{r}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{i_1 \ldots i_m = 1}^{3} \overline{\mathfrak{M}}_{i_1 \ldots i_m} \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} \left(\frac{1}{r}\right)$$

where

$$\overline{\mathfrak{M}}_{i_1\ldots i_m} = \frac{-1}{(n+1)c}\int dV'\, x'_{i_1}\cdots x'_{i_m}\, \vec{\nabla}'\cdot[\vec{r}'\times\vec{J}(\vec{r}')]$$

To make the Cartesian tensors irreducible (the spherical multipoles already are!) we subtract off the traces as in the electrostatic case.

For $\underline{m=0}$ (no indices)

$$\overline{\mathfrak{M}} = \mathfrak{M} = -\frac{1}{c}\int dV'\,\vec{\nabla}'\cdot[\vec{r}'\times\vec{J}(\vec{r}')] = 0$$

no monopole tensor $\Rightarrow$ no magnetic charge

For $\underline{m=1}$

$$\overline{\mathfrak{M}}_i = \mathfrak{M}_i = -\frac{1}{2c}\int dV'\, x'_i\, \vec{\nabla}'\cdot[\vec{r}'\times\vec{J}(\vec{r}')]$$

integrate by parts

$$= +\frac{1}{2c}\int dV'\,[\vec{r}'\times\vec{J}(\vec{r}')]\cdot\underbrace{\vec{\nabla}' x'_i}_{\hat{e}_{x_i}}$$

$$= \frac{1}{2c}\int dV'\,[\vec{r}'\times\vec{J}(\vec{r}')]_i$$

$$\Rightarrow \vec{\mathfrak{M}} = \frac{1}{2c}\int dV'\,\vec{r}'\times\vec{J}(\vec{r}')$$

We now work through the canonical example: Current flowing in a closed one-dimensional loop (wire).

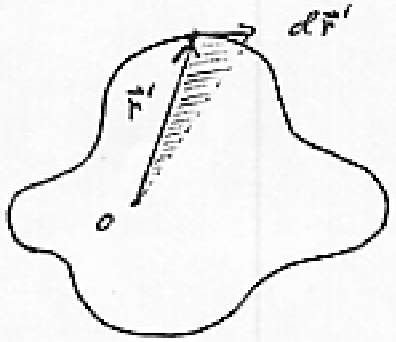
$$\vec{J}(\vec{r}')\,dV' = I\,d\vec{r}'$$

where I is the constant current in the wire = charge per unit time flowing past a point in the wire.

$$\vec{m} = \frac{I}{2c}\oint_C \vec{r}' \times d\vec{r}'$$

The vector $\vec{r}' \times d\vec{r}'$ is normal to the plane containing $\vec{r}'$ and $d\vec{r}'$ so that

$$\vec{r}' \times d\vec{r}' = \hat{n}\, r' |d\vec{r}'| \sin\theta$$



Area of shaded triangle

$$dS = \tfrac{1}{2}\text{ base} \times \text{altitude}$$
$$= \tfrac{1}{2} r' |d\vec{r}'| \sin\theta$$

$$\therefore \quad \vec{m} = \frac{I}{c}\int_{\substack{\text{area of}\\ \text{loop}}} dS\,\hat{n}$$

If the loop is in a plane, then $\hat{n}$ is a constant vector and $\int dS = S =$ area of loop, so that

$$\vec{m} = \frac{I}{c} S\,\hat{n}$$

the direction of $\hat{n}$ is determined by the right-hand rule.

——— End Lecture #20 ———