16 November 95

B) Magnetic Fields in Ponderable Media

Macroscopic matter is composed of molecules. These microscopic entities often possess magnetic dipole moments, due primarily to electrons. To see how this comes about, consider an atom made up of electrons orbiting the nucleus. The magnetic dipole moment is:

$$\vec{m} = \frac{1}{2c}\int dV\, \vec{r} \times \vec{J}(\vec{r})$$

with $\vec{J}(\vec{r}) = \sum_{i=1}^{N} q_i \vec{v}_i \delta^3(\vec{r}-\vec{r}_i) = -e\sum_{i=1}^{N} \vec{v}_i \delta^3(\vec{r}-\vec{r}_i)$

so that

$$\vec{m} = -\frac{e}{2c}\sum_{i=1}^{N} \vec{r}_i \times \vec{v}_i = -\frac{e}{2m_e c}\sum_{i=1}^{N} m_e \vec{r}_i \times \vec{v}_i$$

$\uparrow$ electron mass

$$= -\frac{e}{2mc}\vec{L}$$

where $\vec{L}$ is the total orbital angular momentum of the electrons in the atom. Electrons also possess an intrinsic angular momentum called spin (poorly viewed as the angular momentum of electronic balls of charge rotating about an axis through the center of mass – but with a mass density different from the charge density).

Experiment and Quantum Mechanics (Dirac Equation) give us

$$\vec{m}_{spin} = -g \frac{e}{2mc} \vec{S} \qquad \text{and } g = 2 \text{ is the gyromagnetic ratio}$$

[Actually, when one takes into account the coupling of the electron to the electromagnetic field (quantum electrodynamics - QED) one finds that $g \neq 2$ exactly, but the difference $g-2$ is $\sim 10^{-3}$ and is readily measurable experimentally; the extra bit is called the electron's anomalous magnetic moment.]

$$\left.\frac{g-2}{2}\right|_{experiment} = (11\ 659.22 \pm 0.09) \times 10^{-7}$$

$$\left.\frac{g-2}{2}\right|_{theory} = (11\ 659.19 \pm 0.10) \times 10^{-7}$$

At any rate, the presence of angular momentum generally implies the existence of a magnetic dipole moment.

The magnetic vector potential in the presence of currents due to the <u>macroscopic</u> flow of charge (true current) and the <u>microscopic</u> flow of charge is:

$$\vec{A}(\vec{r}) = \int \frac{dV'}{c} \frac{\vec{J}_{true}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \sum_{\alpha} \int \frac{dV'}{c} \frac{\vec{j}_{\alpha}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\vec{j}_\alpha(\vec{r}\,')$ is the atomic current for the α^{th} atom. This atomic current is only sizeable in a small region about $\vec{r}_\alpha$, so we expand in a Taylor series about $\vec{r}_\alpha$. In the second integral

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{|(\vec{r}-\vec{r}_\alpha)-(\vec{r}\,'-\vec{r}_\alpha)|} = \frac{1}{|\vec{r}-\vec{r}_\alpha|} - (\vec{r}\,'-\vec{r}_\alpha)\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_\alpha|} + \cdots$$

Thus the contribution from the internal currents is

$$\sum_\alpha \int \frac{dV'}{c} \vec{j}_\alpha(\vec{r}\,') \left[\frac{1}{|\vec{r}-\vec{r}_\alpha|} - (\vec{r}\,'-\vec{r}_\alpha)\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_\alpha|} + \cdots \right]$$

The first term above vanishes by an argument we use in the last lecture:

charge is conserved $\Rightarrow \frac{\partial \rho(\vec{r})}{\partial t} = 0 \Rightarrow$ continuity equation becomes $\vec{\nabla}\cdot\vec{j}_\alpha(\vec{r}) = 0 \Rightarrow$ we can write $\vec{j}_\alpha = \vec{\nabla}\times\vec{\xi}(\vec{r})$

then $\int_V \frac{dV'}{c}\vec{j}_\alpha(\vec{r}\,') = \int_S \frac{dS'}{c}\hat{n}\times\vec{\xi}(\vec{r}\,')$ and we take the surface far away from the α^{th} site, where $\vec{j}_\alpha(\vec{r}\,')$ vanishes.

Let's look at the i^{th} component of the second term:

$$-\sum_{\ell=1}^{3}\sum_{\alpha}\int \frac{dV'}{c}\, j_i^{(\alpha)}(\vec{r}')\,(x'_\ell - x_\ell^{(\alpha)})\,\frac{\partial}{\partial x_\ell}\frac{1}{|\vec{r}-\vec{r}^{(\alpha)}|}$$

$$= -\sum_{\ell=1}^{3}\sum_{\alpha}\epsilon_{\ell i k}\, m_k^{(\alpha)}\,\frac{\partial}{\partial x_\ell}\frac{1}{|\vec{r}-\vec{r}^{(\alpha)}|}$$

where $\vec{m}^{(\alpha)} = \frac{1}{2c}\int dV'\,(\vec{r}'-\vec{r}^{\alpha})\times\vec{j}_\alpha(\vec{r}')$

is the magnetic moment of the α^{th} atom. thus

$$\vec{A}(\vec{r}) = \int \frac{dV'}{c}\,\frac{\vec{J}_{true}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \sum_{\alpha}\vec{m}^{(\alpha)}\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}^{\alpha}|}$$

We now replace — as we did in the electrostatic case — the <u>sum</u> over atomic magnetic dipoles by an <u>integral</u> over a continuous density of dipoles:

$$\vec{r}_\alpha \longrightarrow \vec{r}'$$

$$\sum_{\alpha} \longrightarrow \int$$

$$\vec{m}^{(\alpha)} \longrightarrow \vec{M}(\vec{r}')\,dV'$$

$\vec{M}$ is called the <u>magnetization</u>. It is the magnetic analogue of the polarization $\vec{P}$.

The macroscopic magnetic vector potential is:

$$\vec{A}_{macro}(\vec{r}) = \int \frac{dV'}{c} \frac{\vec{J}_{true}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \int dV' \vec{M}(\vec{r}') \times \vec{\nabla} \frac{1}{|\vec{r}-\vec{r}'|}$$

The second term can be rewritten as follows:

change the $\vec{\nabla}$ with respect to $\vec{r}$ into $\vec{\nabla}_{r'}$

$$+ \int dV' \vec{M}(\vec{r}') \times \vec{\nabla}_{r'} \frac{1}{|\vec{r}-\vec{r}'|}$$

then integrate by parts and set the surface term to zero by choosing S outside the bulk material.

$$\vec{A}_{macro}(\vec{r}) = \int \frac{dV'}{c} \frac{\vec{J}_{true}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \int dV' \frac{\vec{\nabla}_{r'} \times \vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\equiv \frac{1}{c} \int dV' \frac{\vec{J}_{total}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

where we define $\vec{J}_{total}(\vec{r}') \equiv \vec{J}_{true}(\vec{r}') + c\vec{\nabla}_{r'} \times \vec{M}(\vec{r}')$.

The second term is called the "magnetization current" density:

$$\vec{J}_{mag}(\vec{r}') = c\, \vec{\nabla}_{r'} \times \vec{M}(\vec{r}')$$

analogous to the polarization charge density.

The macroscopic magnetic field is

$$\vec{B}_{macro}(\vec{r}) = \vec{\nabla} \times \vec{A}_{macro}(\vec{r})$$

This form gives us immediately that $\vec{\nabla} \cdot \vec{B}_{macro}(\vec{r}) = 0$

Also $$\vec{\nabla} \times \vec{B}_{macro}(\vec{r}) = \frac{4\pi}{c} \vec{J}_{total}(\vec{r})$$

$$= \frac{4\pi}{c} \vec{J}_{true}(\vec{r}) + 4\pi \vec{\nabla} \times \vec{M}(\vec{r})$$

or rearranging —

$$\vec{\nabla} \times \left[\vec{B}_{macro}(\vec{r}) - 4\pi \vec{M}(\vec{r}) \right] = \frac{4\pi}{c} \vec{J}_{true}(\vec{r})$$

We define the <u>magnetic intensity</u> $\vec{H}(\vec{r}) = \vec{B}_{macro}(\vec{r}) - 4\pi \vec{M}(\vec{r})$

(What we have been calling the magnetic <u>field</u>, $\vec{B}(\vec{r})$ is called the <u>magnetic induction</u>.) Then

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \frac{4\pi}{c} \vec{J}_{true}(\vec{r})$$

A list of Maxwell's equations in matter for electro-magneto <u>statics</u> is:

$$\vec{\nabla} \times \vec{E}_{macro}(\vec{r}) = 0$$

$$\vec{\nabla} \cdot \vec{D}(\vec{r}) = 4\pi \rho_{true}(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B}_{macro}(\vec{r}) = 0$$

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \frac{4\pi}{c} \vec{J}_{true}(\vec{r})$$

$\vec{D}$ and $\vec{H}$ are macroscopic by definition.

There are also two constitutive relations:

$$\vec{D}(\vec{r}) = \vec{E}_{macro}(\vec{r}) + 4\pi\, \vec{P}(\vec{r})$$

$$\vec{H}(\vec{r}) = \vec{B}_{macro}(\vec{r}) - 4\pi\, \vec{M}(\vec{r})$$

the sign is a matter of convention. We will see the justification soon.

For the special case of linear dielectrics, the first equation above reduces to:

$$\vec{D} = \vec{E} + 4\pi \chi_e \vec{E} = (1 + 4\pi \chi_e)\vec{E} \equiv \epsilon \vec{E}$$

This relation is useful because most materials are para-electric (they polarize in the same direction as the $\vec{E}$ field) and if the field is weak, the linear term is sufficient. Dia-electrics and ferro-electric (electrets) are rare.

In contrast, the three forms of magnetism are all fairly common. We now begin our discussion of dia-magnetism, para-magnetism, and ferro-magnetism and the relation between $\vec{M}$ and $\vec{H}$ for each type.

(i) Diamagnetism

In its simplest form, diamagnetism consists of the following phenomenon: Put a classical charged particle in a uniform magnetic field. The equation of motion derived from the Lorentz force law is:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = q\frac{\vec{v}}{c}\times\vec{B}$$

and the solution is that $\vec{v}$ rotates around $\vec{B}$ with angular velocity $\vec{\Omega} = -\frac{q}{mc}\vec{B}$ (the cyclotron frequency). The velocity and position of the particle are related through:

$$\vec{v} = \vec{\Omega}\times\vec{r} + \vec{v}_0 \qquad (\vec{v}_0 = \text{constant along } \vec{B})$$

If the motion is in a plane perpendicular to $\vec{B}$ then the radius of the circular orbit is

$$r = \frac{v}{|\vec{\Omega}|} \qquad \text{the general motion is a \underline{spiral}.}$$

The angular momentum is

$$\vec{L} = m\vec{r}\times\vec{v} = m\vec{r}\times(\vec{\Omega}\times\vec{r}) = m\vec{\Omega}r^2, \quad \text{thus}$$

$$\vec{L} = -\frac{q}{c}r^2\vec{B}$$

and from before we have

$$\vec{m} = \frac{q}{2mc}\vec{L} = -\frac{q^2}{2mc^2} r^2 \vec{B}$$

Notice that the particle charge q is squared, so that the absolute sign is irrelevant. If we have many particle species with charge q_i and mass m then the net magnetic dipole moment is

$$\vec{m} = -\frac{q^2}{2mc^2}\left(\sum_{i=1}^{N} r_i^2\right)\vec{B}$$

and the magnetic dipole moment per unit volume (the magnetization) is

$$\vec{M} = \frac{\vec{m}}{V} = \underset{\text{dia}}{-}\left[N\frac{q^2}{2mc^2}\langle r^2\rangle\right]\vec{B}$$

N is the number of particles per unit volume. Everything in square brackets is positive. The minus sign between $\vec{M}$ and $\vec{B}$ is characteristic of dia-magnetism.

Thus we see that dia magnetism is universal and results whenever free charge is acted upon by a magnetic field. This is a manifestation of Lenz' Law.

Now we can understand the reason for the sign difference in:

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$
$$H = \vec{B} - 4\pi\vec{M}$$

Most materials are <u>para</u>-electric, and <u>most</u> materials are <u>dia</u>-magnetic. In fact, <u>all</u> materials are <u>dia</u>-magnetic to some extent — in some cases, the <u>para</u>-magnetic effect is larger than the dia.

(ii) Paramagnetism

While diamagnetism is universal, paramagnetism occurs only when molecules possess a magnetic dipole moment (in solids, this is usually due to <u>spin</u> angular momentum as effects of interaction between neighboring molecules). There is competition between the free charges in motion (dia) and the magnetic dipoles (para). If a material is paramagnetic, it is because the paramagnetic susceptibility χ_p is larger than the diamagnetic susceptibility χ_d.

For a <u>weak</u> magnetic field (and for a spin-$\frac{1}{2}$ electron) we obtain:

$$\vec{M} = \chi_p \vec{H} \qquad \text{where} \qquad \chi_p = N \frac{\vec{m}^2}{k_B T} \quad \left(\begin{array}{c}\text{weak}\\\text{field}\end{array}\right)$$

where $\vec{m}^2$ is the square of the molecular magnetic dipole moment, k_B is Boltzmann's constant, and T is the absolute temperature. (This is only the first term in an expansion in $\vec{H}$)

In the large field limit, <u>all</u> of the molecular dipoles align with the applied field and we have reached <u>saturation</u>. The magnetization cannot exceed this value.

$$\vec{M}_{sat} = N |\vec{m}| \, \hat{e} \qquad \text{where } \hat{e} = \frac{\vec{H}}{H} \text{ is the field direction}$$

$$\vec{M} = \chi_p \vec{H} \implies \chi_p \Big|_{sat} = \frac{N|\vec{m}|}{H}$$

(para-magnetic susceptibility at saturation)

The last type is <u>ferro</u>-magnetism.

——— End lecture # 22 ———