This means that there is an outward force per unit area equal to $\sigma^2/2\epsilon_0 = w$ at the surface of the conductor. This result is normally derived by taking the product of the surface-charge density and the electric field, with care taken to eliminate the electric field due to the element of surface-charge density itself.

For a system of n conductors, each with potential V_i and total charge Q_i $(i = 1, 2, \ldots, n)$ in otherwise empty space, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called coefficients of capacity. For a given configuration of the conductors, the linear functional dependence of the potential on the charge density implies that the potential of the ith conductor can be written as

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \qquad (i = 1, 2, \ldots, n)$$

where the p_{ij} depend on the geometry of the conductors. These n equations can be inverted to yield the charge on the ith conductor in terms of all the potentials:

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \qquad (i = 1, 2, \ldots, n) \tag{1.61}$$

The coefficients C_{ii} are called capacities or capacitances while the C_{ij}, $i \neq j$, are called coefficients of induction. The *capacitance of a conductor is therefore the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential.* Sometimes the capacitance of a system of conductors is also defined. For example, the capacitance of two conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor to the potential difference between them. The equations (1.61) can be used to express this capacitance in terms of the coefficients C_{ij}.

The potential energy (1.53) for the system of conductors is

$$W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j \tag{1.62}$$

The expression of the energy in terms of the potentials V_i and the C_{ij}, or in terms of the charges Q_i and the coefficients p_{ij}, permits the application of variational methods to obtain approximate values of capacitances. It can be shown, based on the technique of the next section (see Problems 1.17 and 1.18), that there are variational principles giving upper and lower bounds on C_{ii}. The principles permit estimation with known error of the capacitances of relatively involved configurations of conductors. High-speed computational techniques permit the use of elaborate trial functions involving several parameters. It must be remarked, however, that the need for a Green function satisfying Dirichlet boundary conditions in the lower bound makes the error estimate nontrivial. Further consideration of this technique for calculating capacitances is left to the problems at the end of this and subsequent chapters.

1.12 Variational Approach to the Solution of the Laplace and Poisson Equations

Variational methods play prominent roles in many areas of classical and quantum physics. They provide formal techniques for the derivation of "equations of mo-

tion" and also practical methods for obtaining approximate, but often accurate, solutions to problems not amenable to other approaches. Estimates of resonant frequencies of acoustic resonators and energy eigenvalues of atomic systems come readily to mind.

The far-reaching concept that physical systems in equilibrium have minimal energy content is generalized to the consideration of energy-like functionals. As an example, consider the functional

$$I[\psi] = \frac{1}{2}\int_V \nabla\psi \cdot \nabla\psi \, d^3x - \int_V g\psi \, d^3x \tag{1.63}$$

where the function $\psi(\mathbf{x})$ is well-behaved inside the volume V and on its surface S (which may consist of several separate surfaces), and $g(\mathbf{x})$ is a specified "source" function without singularities within V. We now examine the first-order change in the functional when we change $\psi \to \psi + \delta\psi$, where the modification $\delta\psi(\mathbf{x})$ is infinitesimal within V. The difference $\delta I = I[\psi + \delta\psi] - I[\psi]$ is

$$\delta I = \int_V \nabla\psi \cdot \nabla(\delta\psi) \, d^3x - \int_V g\delta\psi \, d^3x + \cdots \tag{1.64}$$

The neglected term is semipositive definite and is second order in $\delta\psi$. Use of Green's first identity with $\phi = \delta\psi$ and $\psi = \psi$ yields

$$\delta I = \int_V [-\nabla^2\psi - g] \, \delta\psi \, d^3x + \oint_S \delta\psi \frac{\partial\psi}{\partial n} \, da \tag{1.65}$$

Provided $\delta\psi = 0$ on the boundary surface S (so that the surface integral vanishes), the first-order change in $I[\psi]$ vanishes if $\psi(\mathbf{x})$ satisfies

$$\nabla^2\psi = -g \tag{1.66}$$

Recalling that the neglected term in (1.64) is semipositive definite, we see that $I[\psi]$ is a stationary minimum if ψ satisfies a Poisson-like equation within the volume V and the departures $\delta\psi$ vanish on the boundary. With $\psi \to \Phi$ and $g \to \rho/\epsilon_0$, the minimization of the functional yields the "equation of motion" of the electrostatic potential in the presence of a charge density and Dirichlet boundary conditions (Φ given on S and so $\delta\Phi = 0$ there).

The derivation of the Poisson equation from the variational functional is the formal aspect. Equally important, the stationary nature of the extremum of $I[\psi]$ permits a practical approach to an approximate solution for $\psi(\mathbf{x})$. We choose a flexible "trial" function $\psi(\mathbf{x}) = A\Psi(\mathbf{x}, \alpha, \beta, \ldots)$ that depends on a normalization constant A and some number of other parameters, $\alpha, \beta, \ldots$, and is constructed to satisfy the given boundary conditions on the surface S. The function Ψ may be a sum of terms with the parameters as coefficients, or a single function of several parameters; it should be chosen with some eye toward the expected form of the solution. (Intuition plays a role here!) Calculation of $I[\psi]$ gives the function, $I(A, \alpha, \beta, \ldots)$. We now vary the parameters to locate the extremum (actually a minimum) of $I(A, \alpha, \beta, \ldots)$. With the optimum parameters, the trial solution is the best possible approximation to the true solution with the particular functional form chosen. For the Laplace equation, the normalization constant is determined by the Dirichlet boundary values of ψ. For the Poisson equation, it is determined by the source strength $g(\mathbf{x})$, as well as the boundary values on S.

A different functional is necessary for Neumann boundary conditions. Sup-

pose that the boundary conditions on ψ are specified by $\partial\psi/\partial n|_S = f(\mathbf{s})$, where $\mathbf{s}$ locates a point on the surface S. The appropriate functional is

$$I[\psi] = \frac{1}{2}\int_V \nabla\psi \cdot \nabla\psi \, d^3x - \int_V g\psi d^3x - \oint_S f\psi \, da \tag{1.67}$$

The same steps as before with $\psi \to \psi + \delta\psi$ lead to the first-order difference in functionals,

$$\delta I = \int_V [-\nabla^2\psi - g] \, \delta\psi \, d^3x + \oint_S \left(\frac{\partial\psi}{\partial n} - f(\mathbf{s})\right) \delta\psi \, da \tag{1.68}$$

The requirement that δI vanish independent of $\delta\psi$ implies

$$\nabla^2\psi = -g \text{ within } V \qquad \text{and} \qquad \frac{\partial\psi}{\partial n} = f(\mathbf{s}) \text{ on } S \tag{1.69}$$

Again the functional is a stationary minimum for ψ satisfying (1.69). Approximate solutions can be found by the use of trial functions that satisfy the Neumann boundary conditions, just as described above for Dirichlet boundary conditions.

As a simple application to the Poisson equation, consider the two-dimensional problem of a hollow circular cylinder of unit radius centered on the z-axis, with an interior source density $g(\mathbf{x}) = g(\rho)$, azimuthally symmetric and independent of z. The potential vanishes at $\rho = 1$. The "equation of motion" for ψ (a function of ρ alone) in polar coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) = -g(\rho) \tag{1.70}$$

For trial functions we consider finite polynomials in powers of $(1 - \rho)$ and ρ. A three-parameter function of the first type is

$$\Psi_1 = \alpha_1(1 - \rho) + \beta_1(1 - \rho)^2 + \gamma_1(1 - \rho)^3 \tag{1.71}$$

This choice might seem natural because it automatically builds in the boundary condition at $\rho = 1$, but it contains a flaw that makes it a less accurate representation of ψ than the power series in ρ. The reason is that, if the source density g is well behaved and finite at the origin, Gauss's law shows that ψ has a maximum or minimum there with vanishing slope. The requirements at both the origin and $\rho = 1$ are met by a three-parameter trial function in powers of ρ:

$$\Psi_2 = \alpha\rho^2 + \beta\rho^3 + \gamma\rho^4 - (\alpha + \beta + \gamma) \tag{1.72}$$

We expect this trial function in general to be a better approximation to ψ than Ψ_1 for the same number of variational parameters. [We could, of course, impose the constraint, $\alpha_1 + 2\beta_1 + 3\gamma_1 = 0$ on (1.71) to get the proper behavior at the origin, but that would reduce the number of parameters from three to two.]

The functional integral (1.63) for Ψ_2 is easily shown to be

$$\frac{1}{2\pi} I[\Psi_2] = \left[\frac{1}{2}\alpha^2 + \frac{6}{5}\alpha\beta + \frac{4}{3}\alpha\gamma + \frac{3}{4}\beta^2 + \frac{12}{7}\beta\gamma + \gamma^2\right] - [e_2\alpha + e_3\beta + e_4\gamma] \tag{1.73}$$

where $e_n = \int_0^1 g(\rho)(\rho^n - 1)\,\rho\,d\rho$.

The integral for Ψ_1 has the same form as (1.73), but different coefficients. As described above, we seek an extremum of (1.73) by setting the partial derivatives with respect to the parameters α, β, and γ equal to zero. The three coupled algebraic linear equations yield the "best" values,

$$\begin{aligned} \alpha &= 225e_2 - 420e_3 + 210e_4 \\ \beta &= -420e_2 + \frac{2450}{3} e_3 - 420e_4 \\ \gamma &= 210e_2 - 420e_3 + \frac{441}{2} e_4 \end{aligned} \tag{1.74}$$

These values can be inserted into (1.73) to give $I[\Psi_2]_{\min}$ as a not very illuminating function of the e_n. One would then find that the "kinetic" (first) bracket was equal to half the "potential" (second) bracket and opposite in sign, a characteristic of the extremum. **1.13**

To go further we must specify $g(\rho)$. The results for the best trial functions Ψ_1 and Ψ_2 are shown in Fig. 1.9 for the source density,

$$g(\rho) = -5(1 - \rho) + 10^4\rho^5(1 - \rho)^5 \tag{1.75}$$

The choice of source is arbitrary and is chosen to give a potential that is not quite featureless. The "best" parameters for Ψ_2 are $\alpha = 2.915$, $\beta = -7.031$, and $\gamma = 3.642$. The variational integral has the value, $I[\Psi_2]_{\min} = -1.5817$, compared to $I[\psi]_{\text{exact}} = -1.6017$. The fractional error is 1.3%.

Note that the trial function Ψ_1 fails rather badly for $\rho < 0.3$ because it does

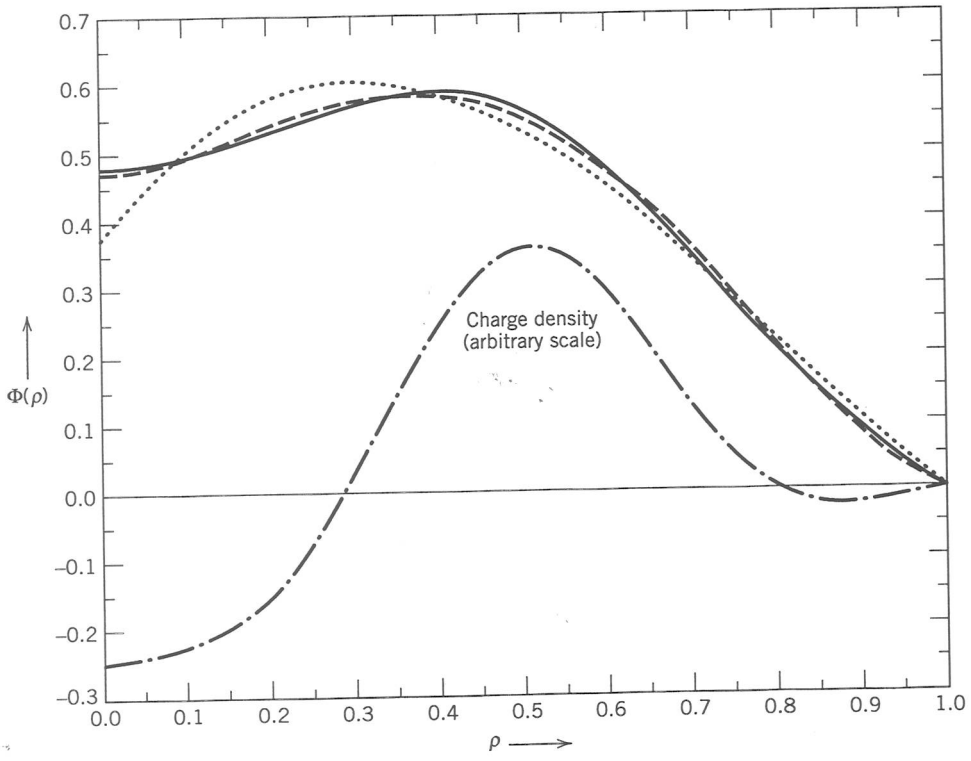

Figure 1.9 Comparison of the exact solution $\psi(\rho)$ (solid curve) with two variational approximations for the potential, Ψ_1 (dotted curve) and Ψ_2 (dashed curve). The charge density (1.75) is indicated by the dash-dot curve (arbitrary scale).

not respect the vanishing slope at $\rho = 0$. Nonetheless, it gives $I[\Psi_1]_{\min} = -1.5136$, which is somewhat, but not greatly, worse than Ψ_2 (5.5% error). The insensitivity of $I[\Psi]$ to errors in the trial function illustrates both a strength and a weakness of the variational method. If the principle is used to estimate eigenvalues (related to the value of $I[\Psi]$), it does well. Used as a method of estimating a solution $\psi \approx \Psi$, it can fail badly, at least in parts of the configuration space.

The reader will recognize from (1.70) that a polynomial source density leads to an exact polynomial solution for ψ, but the idea here is to illustrate the variational method, not to demonstrate a class of explicit solutions. Further illustration is left to the problems at the end of this and later chapters.

1.13 *Relaxation Method for Two-Dimensional Electrostatic Problems*

The relaxation method is an iterative numerical scheme (sometimes called iterative finite difference method) for the solution of the Laplace or Poisson equation in two dimensions. Here we present only its basic ideas and its connection with the variational method. First we consider the Laplace equation with Dirichlet boundary conditions within a two-dimensional region S with a boundary contour C. We imagine the region S spanned by a square lattice with lattice spacing h (and the boundary contour C approximated by a step-like boundary linking lattice sites along C). The independent variables are the integers (i, j) specifying the sites; the dependent variables are the trial values of the potential $\psi(i, j)$ at each site. The potential values on the boundary sites are assumed given.

To establish the variational nature of the method and to specify the iterative scheme, we imagine the functional integral $I[\psi]$ over S as a sum over small domains of area h^2, as shown in Fig. 1.10*a*. We consider the neighboring trial values of the potential as fixed, while the value at the center of the subarea is a variational quantity to be optimized. The spacing is small enough to permit us to approximate the derivatives in, say, the northeast quarter of the subarea by

$$\left(\frac{\partial\psi}{\partial x}\right)_{\text{NE}} = \frac{1}{h}(\psi_{\text{E}} - \psi_0); \qquad \left(\frac{\partial\psi}{\partial y}\right)_{\text{NE}} = \frac{1}{h}(\psi_{\text{N}} - \psi_0)$$

and similarly for the other three quarters. The functional integral over the northeast quarter is

$$\begin{aligned} I_{\text{NE}} &= \frac{1}{2}\int_0^{h/2} dx \int_0^{h/2} dy \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \\ &\approx \frac{1}{8}[(\psi_0 - \psi_{\text{N}})^2 + (\psi_0 - \psi_{\text{E}})^2] \end{aligned} \tag{1.76}$$

The complete integral over the whole (shaded) subarea is evidently

$$I \approx \frac{1}{4}[(\psi_0 - \psi_{\text{N}})^2 + (\psi_0 - \psi_{\text{E}})^2 + (\psi_0 - \psi_{\text{S}})^2 + (\psi_0 - \psi_{\text{W}})^2] \tag{1.77}$$

Minimizing this integral with respect to ψ_0 gives the optimum value,

$$(\psi_0)_{\text{optimum}} = \frac{1}{4}(\psi_{\text{N}} + \psi_{\text{E}} + \psi_{\text{S}} + \psi_{\text{W}}) \tag{1.78}$$