I. Electrostatics

A) Electrostatic Field + Electrostatic Potential

The basic qualitative facts about the electrostatic force are: ① There is one "flavor" of charge and an object may have a positive or negative amount of this charge, and ② Like-sign charges repel while unlike-sign charges attract.

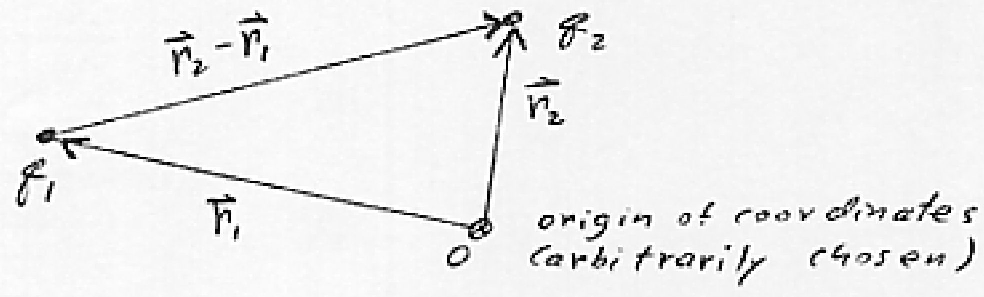
If the charges are idealized to "point charges" then the quantitative statement of the force law is:

$$\vec{F}_{1 \text{ on } 2} = k \frac{q_1 q_2 \; \widehat{r_2 - r_1}}{|r_2 - r_1|^2} = k \frac{q_1 q_2 \; (\vec{r}_2 - \vec{r}_1)}{|r_2 - r_1|^3}$$

This is Coulomb's law, derived from experimental observations that

① The force is proportional to each charge, and
② The force is inversely proportional to the square of the distance between the point charges.

(the inverse square nature, like Newtonian gravity, will imply that the quantum force carrier, the photon, is massless)



$(\vec{r}_2 - \vec{r}_1)$ is independent of the choice of origin; it is the only natural vector in the problem.

The Coulomb force obeys Newton's Third Law:

$$\vec{F}_{1 \text{ on } 2} = - \vec{F}_{21}$$

Not all classical forces obey the Third Law! The magnetic force between two moving charges does <u>NOT</u> obey the Third Law. This uneasy fact leads to trouble with momentum conservation which we will resolve near the end of the course when we take into account the momentum carried by the electromagnetic field itself.

The constant of proportionality <u>k</u> appearing in Coulomb's Law depends on the system of units used.

MKS $\begin{cases} q \text{ in coulombs} \\ r \text{ in meters} \\ F \text{ in Newtons} \end{cases} \Rightarrow k = \frac{1}{4\pi\epsilon_0} \approx 9\times10^9 \frac{N\cdot m^2}{C^2}$

ϵ_0 is the "permittivity of free space"

Gaussian $\begin{cases} q \text{ in statcoulombs} \\ r \text{ in centimeters} \\ F \text{ in dynes} \end{cases} \Rightarrow k = 1$

Jackson uses Gaussian units — so will we.
(well, sometimes.)

RESERVE

Superposition:

The total force produced on one point charge by any number of other point charges is the vector sum of the individual two-body Coulomb's Law forces. We assume that there are no intrinsic many-body forces involving more than pairs of charges.

The force on q at position $\vec{r}$ due to N point charges q_i at $\vec{r}_i$ is:

$$\vec{F}_{\text{on } q} = \sum_{i=1}^{N} \frac{k q q_i (\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} \equiv q \vec{E}(\vec{r})$$

where we have introduced the <u>Electrostatic Field</u> at position $\vec{r}$

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{k q_i (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

In the sum, i is a dummy variable. Once summed over, it cannot appear on the left hand side.

$\vec{E}(\vec{r})$ is a <u>vector</u> field.

As a shorthand, we will call this simply the Electric Field.

Sometimes, we wish to consider <u>continuous</u> distributions of charge. The charge contained in an infinitesimal volume element dV' located at $\vec{r}'$ is:

$$dq = dV' \rho(\vec{r}')$$

where $\rho(\vec{r}')$ is the electric charge density at $\vec{r}'$. $\rho(\vec{r}')$ is a <u>scalar</u> field.

The electrostatic field at position $\vec{r}$ is given by:

$$\vec{E}(\vec{r}) = \int_V \frac{k\,dq\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \int_V dV' k\rho(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$\vec{r}\,'$ is the (dummy) integration variable. Once integrated over, it cannot appear on the lefthand side. $\vec{r}\,'$ is also called the "source point" since that is where the charge, the source of electric field, sits.

$\vec{r}$ is called the "field point" since that is the location at which we choose to measure the electric field.

Sometimes, we will consider <u>surface</u> or <u>line</u> distributions of charge rather than volume distributions.

$$\vec{E}(\vec{r}) = \int_S dS' k\sigma(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

surface S

surface charge density

$$\vec{E}(\vec{r}) = \int_C dl' k\lambda(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Curve C

linear charge density

We can recover the discrete formulae from the continuous formulae by writing the volume charge density for a point charge as

$$\rho(\vec{r}\,') = q_1\, \delta^3(\vec{r}\,' - \vec{r}_1)$$

where the point charge has magnitude q_1 and sits at location $\vec{r}_1$.

$\delta^3(\vec{r}\,' - \vec{r}_1)$ is the three-dimensional <u>Dirac delta function</u>, which is not a function at all. It is called a "generalized function" or a "distribution" and it is only one of many such objects.

$$\delta^3(\vec{r}\,' - \vec{r}_1) \equiv \delta(x' - x_1)\, \delta(y' - y_1)\, \delta(z' - z_1)$$

Properties:

① $\delta^3(\vec{r} - \vec{r}\,') = 0$ if $\vec{r} \neq \vec{r}\,'$

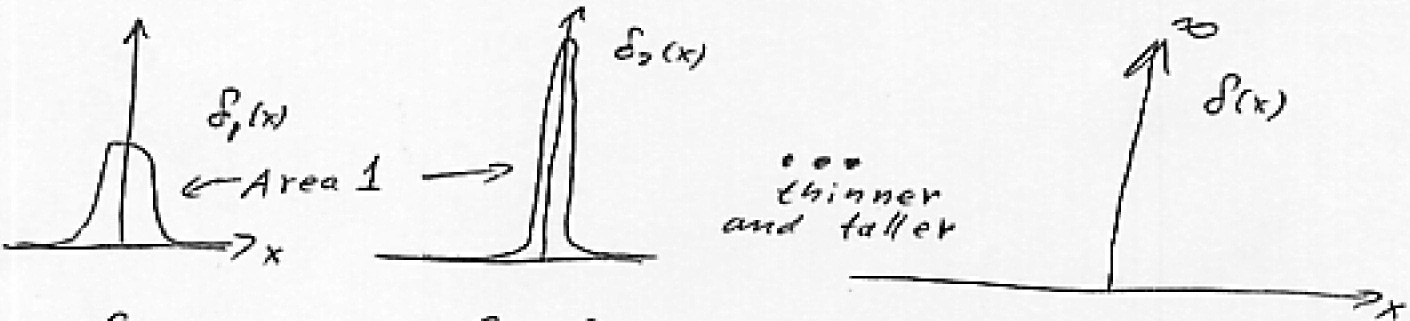
② $\displaystyle\int_V dV'\, \delta^3(\vec{r} - \vec{r}\,') = \begin{cases} 1 & \text{if } V \text{ includes } \vec{r} \\ 0 & \text{if } V \text{ does not} \end{cases}$

①+② $\Rightarrow \displaystyle\int dV'\, f(\vec{r}\,')\, \delta^3(\vec{r} - \vec{r}\,') = f(\vec{r})$

For many point charges q_i at $\vec{r}_i$, the charge density is

$$\rho(\vec{r}\,') = \sum_i q_i\, \delta^3(\vec{r}\,' - \vec{r}_i)$$

You can think of the delta function as a series of approximations:



$$\int dx\, \delta_1(x) = 1 = \int dx\, \delta_2(x)$$

$$\int \delta(x)\, dx = 1$$

zero width, but infinite height with finite area = 1

You are now <u>prepared to solve Problem # 1.</u>

Note a useful mathematical relation that will simplify calculation of $\vec{E}(\vec{r})$:

$$\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = -\vec{\nabla}_r \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right)$$

Problem #2: Prove this

gradient with respect to unprimed variables.

The Electric field can now be written as:

$$\vec{E}(\vec{r}) = -\int_V dV' k \rho(\vec{r}\,') \vec{\nabla}_r \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \equiv -\vec{\nabla} \Phi(\vec{r})$$

where $\Phi(\vec{r}) = \int_V dV' k \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$ is the <u>Electrostatic Potential</u>

$\Phi(\vec{r})$ is a <u>scalar</u> field

The gradient operation with respect to unprimed variables can be brought outside the integral above since this operation has no effect on the primed integration variables.

$\Phi(\vec{r})$ is easier, in general, to calculate than $\vec{E}(\vec{r})$. $\vec{E}(\vec{r})$ can then be obtained from $\Phi(\vec{r})$.

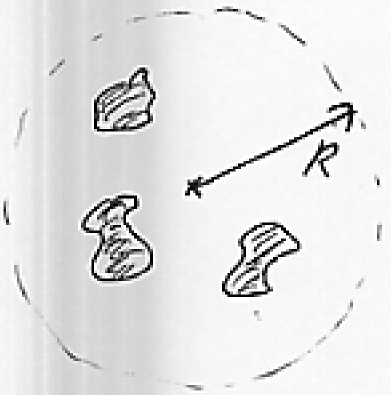
If we know $\rho(\vec{r}\,')$ <u>everywhere</u>, then we could find $\Phi(\vec{r})$ and hence $\vec{E}(\vec{r})$ (analytically or numerically by computer). The electrostatics part of the course would be over!

The problem is that very often we do <u>not</u> know $\rho(\vec{r}')$ everywhere. Typically, we know $\rho(\vec{r}')$ in some region and that charge results in the appearance of a charge distribution elsewhere, e.g. on conducting surfaces. The "induced" charge density is in general <u>very</u> difficult to determine. More on this later.

For now, we assume that $\rho(\vec{r}')$ is specified completely.

<u>Multipole Expansion</u>

Suppose $\rho(\vec{r}')$ is known everywhere and is confined to a sphere of radius R



Calculate the potential $\Phi(\vec{r})$ (and field) outside R, usually <u>far</u> outside.

Simple Examples

i) single point charge q

Choose the origin of coordinates at q.

$$\rho(\vec{r}') = q\,\delta^3(\vec{r}'-0) = q\,\delta^3(\vec{r}')$$

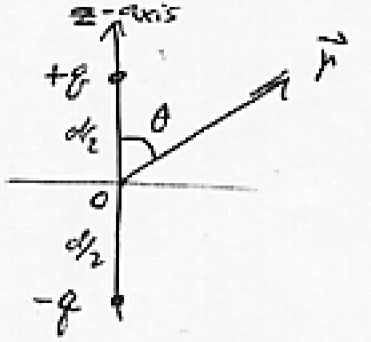
$$\Phi(\vec{r}) = \int_V dV' k\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = q\int_V dV' \frac{k\delta^3(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{kq}{|\vec{r}-\vec{0}|} = \frac{kq}{r}$$

"Monopole" term

Typically, the expansion contains several term, the zeroth of which is the monopole term. In this simple example that's the only term because we have placed a "monopole" at the origin.

ii) Two equal but opposite point charges separated by a distance d.

Choose a symmetric problem - put the origin between the charges:



$$\rho(\vec{r}\,') = q\,\delta^3(\vec{r}\,' - \tfrac{d}{2}\vec{e}_3) - q\,\delta^3(\vec{r}\,' + \tfrac{d}{2}\vec{e}_3)$$

$$\Phi(\vec{r}) = \int_V dV' k \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \frac{kq}{|\vec{r} - \frac{d}{2}\vec{e}_3|} - \frac{kq}{|\vec{r} + \frac{d}{2}\vec{e}_3|}$$

If $r \gg \frac{d}{2}$, then a power series expansion in $\frac{d}{2r}$ will converge. The <u>sphere</u> which contains all the charge has radius $R = \frac{d}{2}$

(Incidentally, <u>any</u> sphere is valid in the first example. <u>Any</u> sphere centered on the origin contains all the charge, so the expansion converges. <u>Of course</u> it converges; there is only one term!)

Let's expand

$$\frac{1}{|\vec{r} \mp \frac{d}{2}\vec{e}_3|} = \frac{1}{\sqrt{(\vec{r} \mp \frac{d}{2}\vec{e}_3)^2}} = \frac{1}{\sqrt{r^2 \mp rd\cos\theta + \frac{d^2}{4}}}$$

$$= \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2}$$

use binomial expansion

$$= \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta + \text{higher order}\right)$$

sign change!

RESERVE

$$\Phi(\vec{r}) = \frac{kq}{r}\left[\frac{d}{r}\cos\theta + \mathcal{O}\left(\frac{d^3}{r^3}\right)\right]$$

leading non-zero term, not monopole, but <u>dipole</u>

<u>Note</u>: Only odd powers of $\left(\frac{d}{r}\right)$ appear. This is an "accident" of the symmetry of the problem.

— End lecture #1 —