

28 November 95

III. Time-Varying Electromagnetic Fields

We can not speak about time-varying electric fields alone, or time-varying magnetic fields alone.

A) Ohm's Law

A conservative electric field cannot maintain a steady current in a medium of finite conductivity.

To see this we construct a simple model of current flow, which will obey Ohm's Law.

Imagine a collection of charged particles which are accelerated by an electric field and which scatter from an obstacle after an average time τ . The charged particles then accelerate anew.

We model the average scattering with a viscous force

$$-\frac{m}{\tau} \vec{v} \quad \tau \text{ is the "mean free time" between scatterings}$$

The force on one charged particle is:

$$\vec{F} = m\vec{a} = m \frac{d\vec{v}}{dt} = q\vec{E} - \frac{m}{\tau} \vec{v}$$

In the steady state, $\vec{a} = 0$ and therefore $\vec{v} = \text{constant}$

$$0 = q\vec{E} - \frac{m}{\tau} \vec{v} \Rightarrow \vec{v} = \left(\frac{\tau}{m} q\right) \vec{E} = \mu \vec{E}$$

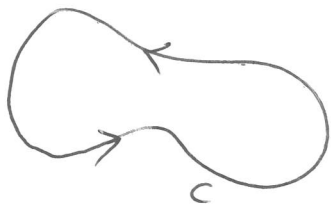
the combination $\frac{\tau}{m} q$ is called the mobility μ .

The current density $\vec{J} = q \vec{v} = N q \vec{v}$

where N is the number density of charges per unit volume.

$$\vec{J} = N q \vec{v} = \left(N q^2 \frac{\tau}{m}\right) \vec{E} = \sigma \vec{E} \quad \left\{ \begin{array}{l} \text{this is} \\ \text{ohm's Law} \end{array} \right.$$

σ is the conductivity = $\frac{1}{\text{resistivity}}$



choose the mathematical path C to coincide with the real circuit. Then if the electric field is conservative:

$$0 = \oint_C d\vec{\ell} \cdot \vec{E}_{\text{conservative}} = \oint_C d\vec{\ell} \cdot \frac{\vec{J}}{\sigma} = \frac{qJ}{\sigma}$$

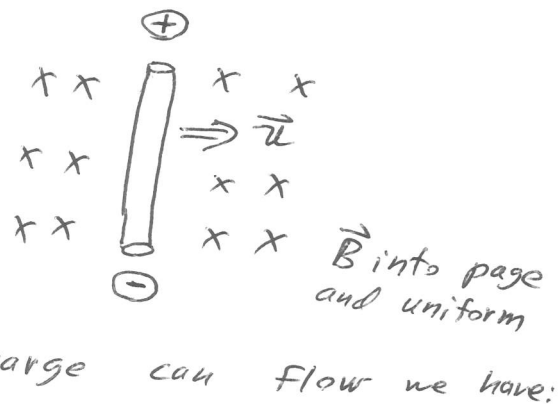
where the last equality follows from the fact that \vec{J} and $d\vec{\ell}$ are parallel.

So either 1) $\sigma = \infty$ the material is a superconductor
 or 2) $J = 0$ no current flows. ■

So what does keep current flowing in a resistive circuit if not a conservative electric field?

Electromotive Force (EMF) symbol = \mathcal{E}

Consider a rod of length l moving in a uniform magnetic field \vec{B} . Charge builds up at the ends of the rods and creates an electric field in the rod. When no more charge can flow we have:



$$\vec{F} = 0 = q \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right)$$

so $\vec{E} = -\frac{\vec{u}}{c} \times \vec{B}$ but in the wire only!

choose a closed curve C that includes the rod and closes outside the rod.



We know

$$1) \oint_C d\vec{l} \cdot \vec{E} = 0$$

since \vec{E} is static and therefore conservative,

$$2) \frac{\vec{u}}{c} \times \vec{B} \text{ is non-zero only in the rod.}$$

The EMF is defined to be

$$\oint_C d\vec{l} \cdot \left[\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right] \equiv \mathcal{E} = \oint_C d\vec{l} \cdot \left(\frac{\vec{u}}{c} \times \vec{B} \right) = \left(\frac{\vec{u}}{c} \times \vec{B} \right) \cdot \vec{l}$$

where $|\vec{l}|$ is the length of the rod.

Now consider a closed wire loop that moves in an inhomogeneous (changes in space) but static (time-independent) magnetic field,



$$\mathcal{E} = \oint_C d\vec{\ell} \cdot \left(\frac{\vec{u}}{c} \times \vec{B} \right) \quad \text{use Stoke's Theorem}$$

$$= \frac{1}{c} \int_S \hat{n} \cdot \vec{\nabla} \times (\vec{u} \times \vec{B})$$

\mathcal{R} any open surface bounded by C

$$\text{But } \vec{\nabla} \times (\vec{u} \times \vec{B}) = \vec{u} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{u}) + (\vec{B} \cdot \vec{\nabla}) \vec{u} - (\vec{u} \cdot \vec{\nabla}) \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{by a Maxwell Equation}$$

and \vec{u} is independent of space (all points in the wire move with same velocity)

$$\vec{\nabla} \cdot \vec{u} = 0 \quad \text{and} \quad (\vec{B} \cdot \vec{\nabla}) \vec{u} = 0$$

$$\therefore \vec{\nabla} \times (\vec{u} \times \vec{B}) = - (\vec{u} \cdot \vec{\nabla}) \vec{B}$$

A little digression into the convective derivative:

If a function $\vec{\Psi}(\vec{r}, t)$ depends on space and time then the total time derivative is

$$\frac{d}{dt} \vec{\Psi}(\vec{r}, t) = \frac{\partial}{\partial t} \vec{\Psi}(\vec{r}, t) + \sum_{i=1}^3 \frac{\partial x_i}{\partial t} \frac{\partial}{\partial x_i} \vec{\Psi}(\vec{r}, t)$$

$$= \frac{\partial}{\partial t} \vec{\Psi}(\vec{r}, t) + (\vec{u} \cdot \vec{\nabla}) \vec{\Psi}(\vec{r}, t)$$

The first term represents changes in $\vec{\Psi}(\vec{r}, t)$ due to explicit time dependence. The second term represents changes in $\vec{\Psi}(\vec{r}, t)$ due to motion, because the coordinates change in time $\vec{r} = \vec{r}(t)$.

Our magnetic field is inhomogeneous, but static

$$\vec{B} = \vec{B}(\vec{r}) \quad \text{so} \quad \frac{d}{dt} \vec{B}(\vec{r}) = (\vec{u} \cdot \vec{\nabla}) \vec{B}(\vec{r})$$

there is no partial time derivative term.

Putting all this together:

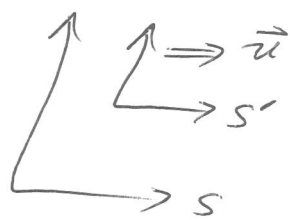
$$\mathcal{E} = \frac{1}{c} \int_S dS \hat{n} \cdot \vec{\nabla} \times (\vec{u} \times \vec{B}) = -\frac{1}{c} \int_S dS \hat{n} \cdot \frac{d\vec{B}}{dt} \equiv -\frac{1}{c} \frac{d\Phi_B}{dt}$$

$\Phi_B \equiv \int_S dS \hat{n} \cdot \vec{B}$ is the magnetic flux through the open surface S .

Galilean Transformation of the Electric Field

In some sense, special relativity is "built into" electromagnetism, in fact it was electromagnetism that led Einstein to formulate his theory.

Consider two inertial frames: S' moves with velocity \vec{u} relative to frame S .



$$\left. \begin{aligned} \vec{r} &= \vec{r}' + \vec{u}t \\ \vec{v} &= \vec{v}' + \vec{u} \\ \vec{a} &= \vec{a}' \\ \vec{F} &= \vec{F}' \end{aligned} \right\} \text{Galilean transformation}$$

The Lorentz force is:

$$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) = \vec{F}' = q(\vec{E}' + \frac{\vec{v}'}{c} \times \vec{B}')$$

\vec{u} is the frame velocity, \vec{v} is the particle velocity in S .

This relation holds for all velocities $v \ll c$, in particular for $\vec{v} = 0$. Then

$$\vec{E} = \vec{E}' - \frac{\vec{u}}{c} \times \vec{B}'$$

This implies

$$\frac{\vec{v}}{c} \times \vec{B} = \frac{\vec{v}}{c} \times \vec{B}' \quad \text{but one should not conclude}$$

that $\vec{B} = \vec{B}'$, they differ by a correction term of order $(\frac{v}{c})$.

The corrections to the Lorentz force are then $\mathcal{O}(\frac{v^2}{c^2})$.

We are now ready to derive Faraday's Law.

$$\begin{aligned}\mathcal{E} &= \oint_C \vec{dl} \cdot (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) = -\frac{1}{c} \frac{d}{dt} \Phi_B \\ &= \oint_C \vec{dl} \cdot \vec{E}' = -\frac{1}{c} \frac{d}{dt} \int_S \vec{ds} \cdot \vec{B} \\ &= \int_S \vec{ds} \cdot \vec{\nabla} \times \vec{E}' = -\frac{1}{c} \frac{d}{dt} \int_S \vec{ds} \cdot \vec{B}' + \mathcal{O}\left(\frac{v^2}{c^2}\right)\end{aligned}$$

Choose the path C at rest in S' .

\vec{B} depends on \vec{r} , not on time, \vec{B}' is fixed in S' but changes in time.

$$\frac{d}{dt} \vec{B}' = \frac{\partial}{\partial t} \vec{B}' \quad \text{the convective term vanishes}$$

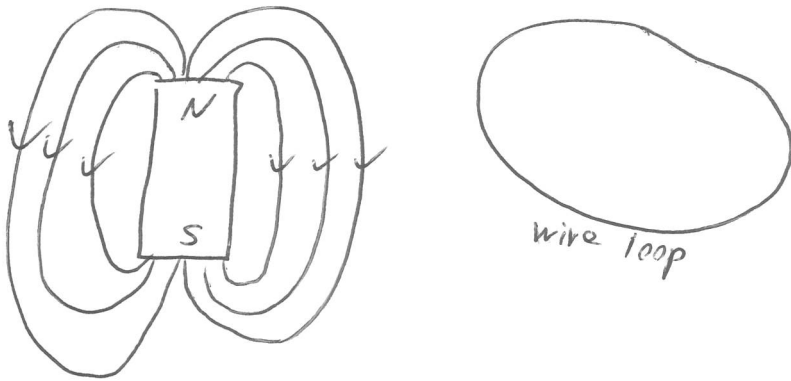
$$\int_S \vec{ds} \cdot \vec{\nabla} \times \vec{E}' = \int_S \vec{ds} \cdot \left(-\frac{1}{c} \frac{\partial}{\partial t} \vec{B}'\right)$$

This must hold for any surface S

$$\Rightarrow \vec{\nabla} \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{B}'}{\partial t} \quad \text{This is Faraday's Law}$$

A time-varying magnetic field generates a non-conservative electric field.

Faraday's Law



If we move the wire loop through the time-independent magnetic field of the stationary magnet, an EMF will be induced in the loop causing current to flow. Our intuition and Galilean relativity would seem to suggest that an EMF will also be induced if we hold the loop stationary and move the magnet, but that is a much different situation from the point of view of the descriptive equations. There is now a time-dependent magnetic field. Indeed, the same EMF is induced, but its source is the changing magnetic flux due to the changing magnetic field.

Displacement Current

Let's look at Maxwell's Equations so far:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 4\pi \rho_e & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{B} &= \frac{4\pi}{c} \vec{J}_e\end{aligned}$$

where we have subscripted ρ_e and \vec{J}_e to represent electric charge and current densities in anticipation of the next section.

There is one apparent asymmetry and one real asymmetry in the equations above,

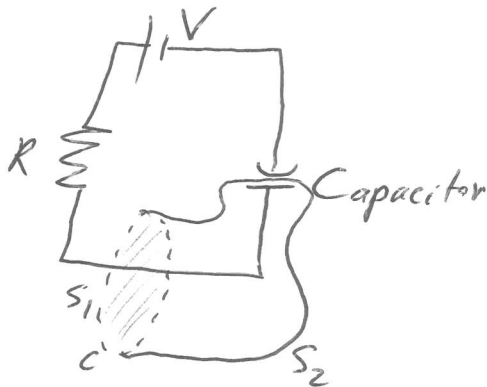
The apparent asymmetry can be lifted by introducing magnetic charge (ρ_m) and magnetic current (\vec{J}_m) densities. If particle physicists discovered magnetic monopoles, Maxwell's Equations would survive with the following modifications:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 4\pi \rho_e & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} - \frac{4\pi}{c} \vec{J}_m \\ \vec{\nabla} \cdot \vec{B} &= 4\pi \rho_m & \vec{\nabla} \times \vec{B} &= \underline{???} + \frac{4\pi}{c} \vec{J}_e\end{aligned}$$

Now the real asymmetry is manifest.

Maxwell noticed this asymmetry and used a thought experiment involving a parallel-plate capacitor to show the discrepancy.

Consider the following circuit:



The resistor is present to limit the current.

When the circuit is first connected, a current flows then decays exponentially.

We use Ampere's law in integral form:

$$\oint_C d\vec{l} \cdot \vec{B} = \frac{4\pi}{c} I_{\text{enclosed}} = \begin{cases} \frac{4\pi}{c} I & \text{through } S_1 \\ 0 & \text{through } S_2 \end{cases}$$

where the surface S_2 goes through the gap in the capacitor. No current flows through S_2 .

Let's see what happens if we add the term demanded by symmetry:

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J}_e$$

or in integral form:

$$\oint_C d\vec{\ell} \cdot \vec{B} = \frac{1}{c} \frac{\partial \Phi_e}{\partial t} + \frac{4\pi}{c} I_{\text{enclosed}}$$

where Φ_e is the electric flux through S

$$\Phi_e = \int_S ds \hat{n} \cdot \vec{E}$$

For a parallel-plate capacitor:

$$E = \frac{4\pi Q}{A} \quad \Phi_e = EA = 4\pi Q$$

$$\frac{\partial \Phi_e}{\partial t} = 4\pi \frac{\partial Q}{\partial t} = 4\pi I$$

$$\frac{1}{c} \frac{\partial \Phi_e}{\partial t} = \frac{4\pi}{c} I \quad \text{through } S_2$$

Now there is no discrepancy. $\oint_C d\vec{\ell} \cdot \vec{B}$ gives the same answer no matter which surface is chosen.

Maxwell called the term $\frac{1}{4\pi} \frac{\partial \Phi_e}{\partial t}$ the displacement current, but realize that it is not a real current at all.

Maxwell added this term for mathematical reasons only - the effect was too small to measure experimentally at the time. In his honor

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J}_e$$

is called the Ampere - Maxwell Law.

This term is absolutely necessary for wave propagation. Maxwell was trying to describe light as an electromagnetic wave.

An alternate derivation:

Until now, we have considered only steady currents for which charge conservation gives (from the continuity equation)

$$\vec{\nabla} \cdot \vec{J} = 0$$

For time-varying currents, charges can accumulate in regions, so we have in general

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Without Maxwell's displacement current,
the set of four equations are inconsistent.

$$0 = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \frac{4\pi}{c} \vec{\nabla} \cdot \vec{J} \neq 0 \quad (\text{in general})$$

With the modification

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

we have

$$\begin{aligned} 0 = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) &= \frac{4\pi}{c} \vec{\nabla} \cdot \vec{J} + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) \\ &= \frac{4\pi}{c} \left[\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right] = 0 \end{aligned}$$

In a medium of permittivity ϵ and permeability μ , Maxwell's Equations for the macroscopic fields are:

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho_{\text{true}} \quad \vec{\nabla} \times \vec{E}_{\text{macro}} = -\frac{1}{c} \frac{\partial \vec{B}_{\text{macro}}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B}_{\text{macro}} = 0 \quad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J}_{\text{true}} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

along with the constitutive equations:

$$\vec{D} = \vec{E}_{\text{macro}} + 4\pi \vec{P}$$

$$\vec{H} = \vec{B}_{\text{macro}} - 4\pi \vec{M}$$