Two answers to the "hidden momentum" puzzle:

① Footnote 33 Babson et al Am J Phys 77(9) Sep 2009

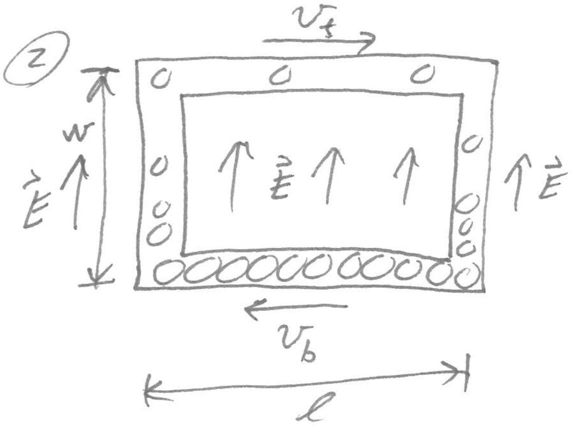
The center of energy is moving from the battery end to the resistor end and the momentum in the $\vec{E}$ and $\vec{B}$ fields is the momentum associated with this motion.

Remember Maxwell's Equations are already relativistic.

Newton's Laws must be fixed:

center of mass $\longrightarrow$ center of energy.

②



N_t charges on top, N_b on bottom

Current $I = \lambda v$ same in all four sides.

$$I = \frac{q N_t v_t}{\ell} = \frac{q N_b v_b}{\ell}$$

Classical momentum: $\vec{P}_{class} = m N_t v_t - m N_b v_b = \frac{mI\ell}{q} - \frac{mI\ell}{q} = 0$

Relativistic momentum:

$$\vec{P}_{rel} = m\gamma_t N_t v_t - m\gamma_b N_b v_b = \frac{mI\ell}{q}(\gamma_t - \gamma_b) \neq 0$$

the top segment electrons are moving (slightly) faster.

$\gamma_t mc^2 - \gamma_b mc^2 = qEw = qV$ energy supplied

Field Angular Momentum

Recall: $\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B}$ the Poynting vector is the energy per unit time per unit area transported by the fields. The physically measurable quantity is $\oint\!\!\oint_S \vec{S} \cdot d\vec{a}$.

Recall also: the Field Linear Momentum is

$$\vec{P}_{em} = \mu_0 \epsilon_0 \iiint_V \vec{S}(\vec{r})\, dV = \epsilon_0 \iiint_V \vec{E}(\vec{r}) \times \vec{B}(\vec{r})\, dV$$

The Field Angular Momentum is

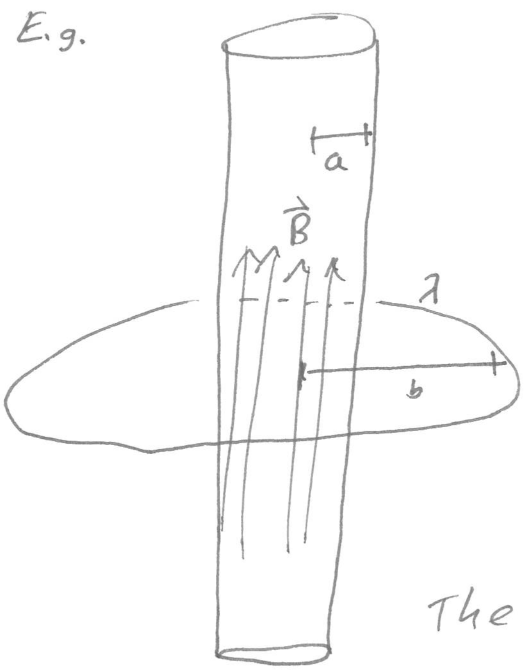
$$\vec{L}_{em} = \mu_0 \epsilon_0 \iiint_V \vec{r} \times \vec{S}(\vec{r})\, dV$$

$$= \epsilon_0 \iiint_V \vec{r} \times \left[\vec{E}(\vec{r}) \times \vec{B}(\vec{r})\right] dV$$

Radiometer demonstration.

Feynman disk paradox.

E.g.



A ring with linear charge density λ and radius b is suspended outside a solenoid of radius $a<b$ with field $\vec{B}$ inside. What happens when the field is turned off?

The changing $\vec{B}$ field will induce an $\vec{E}$ field in the $\hat{\varphi}$ direction which will cause the charged wheel to spin. (Faraday's Law)

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_s \vec{B}\cdot d\vec{a} = -\frac{d}{dt}\Phi_B = -\pi a^2\frac{dB}{dt}$$

Torque on ring

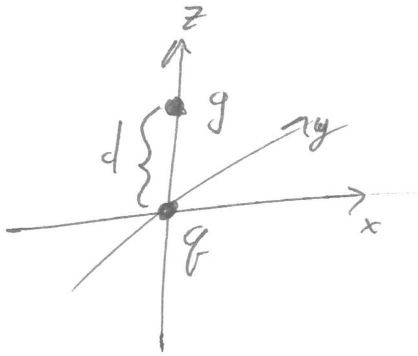
$$N = b\lambda\oint_c E\,dl = -b\lambda\pi a^2\frac{dB}{dt}$$

Angular momentum imparted to ring when $\vec{B}\to 0$

$$L = \int N\,dt = \lambda b\pi a^2 B.$$

This angular momentum came from the $\vec{E}+\vec{B}$ fields.

E.g. One electric monopole q at the origin, one magnetic monopole g at $d\hat{z}$.



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0} \frac{q\vec{r}}{r^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{g(\vec{r}-d\hat{z})}{|\vec{r}-d\hat{z}|^3}$$

$$= \frac{\mu_0 g}{4\pi} \frac{(\vec{r}-d\hat{z})}{(r^2+d^2-2rd\cos\theta)^{3/2}}$$

Linear Momentum Density: $\quad (\vec{r}\times\vec{r}=0) \quad \vec{P}_{em} = \iiint \vec{\wp}(\vec{r})\,dV$

$$\vec{\wp}(\vec{r}) = \epsilon_0(\vec{E}\times\vec{B}) = \frac{\mu_0 q g}{(4\pi)^2} \frac{-d(\vec{r}\times\hat{z})}{r^3(r^2+d^2-2rd\cos\theta)^{3/2}}$$

Angular Momentum Density: $\quad \vec{L}_{em} = \iiint \vec{\mathcal{L}}(\vec{r})\,dV$

$$\vec{\mathcal{L}}(\vec{r}) = \vec{r}\times\vec{\wp}(\vec{r}) = -\frac{\mu_0 q g d}{(4\pi)^2} \frac{(r^2\cos\theta\,\hat{r} - r^2\hat{z})}{r^3(r^2+d^2-2rd\cos\theta)^{3/2}}$$

$\downarrow$ homework

$$\vec{L}_{em} = \frac{\mu_0 q g}{4\pi}\hat{z}$$ independent of separation $\underline{d}$.

Aside In Quantum Mechanics, angular momentum is quantized in units of $\frac{\hbar}{2}$

$$\Rightarrow \quad \frac{\mu_0 q g}{4\pi} = n\frac{\hbar}{2} \qquad n = 1, 2, 3, \ldots$$

P.A.M. Dirac - if even one magnetic monopole $\underline{g}$ exists somewhere in the Universe, then electric charge would be quantized as observed.

Electromagnetic Waves in Vacuum

Maxwell's Equations:

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

coupled first-order partial differential equations.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{0}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$ — wave equation

$$\mu_0\epsilon_0 = \frac{1}{c^2}$$

c = speed of propagation

similarly

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0$$

uncoupled second-order partial differential equations.

Because Maxwell's Equations are linear, it is convenient to work with complex solutions (easy to take derivatives) and then take the real (or imaginary – but the convention is real) part to be the <u>physical</u> solution.

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right]$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 \exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right]$$

$\uparrow$ complex

$\vec{k}$ = wave vector

ω = angular frequency

$$\left.\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = 0 \implies i\vec{k}\cdot\vec{E}_0 = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \implies i\vec{k}\cdot\vec{B}_0 = 0 \end{array}\right\}$$ the waves are <u>transverse</u>

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \implies i\vec{k}\times\vec{E}_0 = i\omega\vec{B}_0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \implies i\vec{k}\times\vec{B}_0 = -\mu_0\epsilon_0 i\omega\vec{E}_0 = -\frac{i\omega\vec{E}_0}{c^2}$$

Combining these and using $\vec{k}\cdot\vec{E}_0 = 0$:

$$\vec{k}\times(\vec{k}\times\vec{E}_0) = -k^2\vec{E}_0 = -\frac{\omega^2}{c^2}\vec{E}_0$$

$$\implies |\vec{k}| = \frac{\omega}{c} = \frac{1}{\lambda}$$ $\leftarrow$ wavelength

Electromagnetic Waves in Matter

Plane waves in a permeable dielectric medium.

Maxwell's Equations with no sources: $\rho_{free} = 0$, $\vec{J}_{free} = 0$

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{D} = \vec{D}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{B} = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{H} = \vec{H}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{D}_0 = \epsilon\vec{E}_0 \qquad \vec{H}_0 = \frac{\vec{B}_0}{\mu}$$

Assume the medium is linear and homogeneous.

ϵ and μ will not depend on $\hat{k}$, but they can depend on ω: $\epsilon(\omega)$, $\mu(\omega)$ which leads to dispersion. $\vec{\nabla}\cdot\vec{D} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$ gives:

$\vec{k}\cdot\vec{E}_0 = 0$, $\vec{k}\cdot\vec{B}_0 = 0$ waves are transverse.

$|\vec{k}| = \frac{\omega}{v}$ where $v \equiv \frac{c}{n} = \frac{1}{\sqrt{\epsilon\mu}}$ phase velocity $v(\omega)$

$n(\omega) = \sqrt{\frac{\epsilon(\omega)\mu(\omega)}{\epsilon_0\mu_0}}$ is the index of refraction

If $\mu \simeq \mu_0$ then $n \simeq \sqrt{\epsilon_{rel}}$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies i\vec{k}\times\vec{E}_0 = i\omega\vec{B}_0$$

Once the angular frequency ω, the direction of propagation $\hat{k}$, and the amplitude of the electric field E_0 (complex, remember) are given, then all aspects of the wave are determined.

$$k = |\vec{k}| = \frac{\omega\sqrt{\epsilon(\omega)\mu(\omega)}}{c}$$

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega}$$

The physical wave is then

$$\vec{\mathcal{E}}(\vec{r},t) = R_e\left[\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\vec{\mathcal{B}}(\vec{r},t) = R_e\left[\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

We also introduce polarization unit vectors $\vec{\epsilon}_1, \vec{\epsilon}_2$

with $\vec{\epsilon}_i\cdot\vec{\epsilon}_j = \delta_{ij} \quad (i,j=1,2)$

$$\vec{\epsilon}_1\times\vec{\epsilon}_2 = \hat{k}$$

$$\vec{\epsilon}_i\cdot\hat{k} = 0 \qquad (i=1,2)$$

Then $$\vec{\mathcal{E}}(\vec{r},t) = \sum_{i=1}^{2} \vec{\epsilon}_i \, \mathcal{E}_i(\vec{r},t)$$

$$\mathcal{E}_i(\vec{r},t) = \vec{\epsilon}_i \cdot \vec{\mathcal{E}}(\vec{r},t) = Re\left[\vec{\epsilon}_i \cdot \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$= Re\left[E_{0i} \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$E_{0i} = \vec{\epsilon}_i \cdot \vec{E}_0 = |E_{0i}| e^{i\varphi_i}$$

$$\mathcal{E}_i(\vec{r},t) = |E_{0i}| \cos(\vec{k}\cdot\vec{r}-\omega t + \varphi_i)$$

$$= |E_{0i}| \left[\cos\varphi_i \cos(\vec{k}\cdot\vec{r}-\omega t) - \sin\varphi_i \sin(\vec{k}\cdot\vec{r}-\omega t)\right]$$

These two equations $(i=1,2)$ can be solved for $\cos(\vec{k}\cdot\vec{r}-\omega t)$ and $\sin(\vec{k}\cdot\vec{r}-\omega t)$ in terms of φ_1, φ_2, $|E_{01}|$, $|E_{02}|$, $\mathcal{E}_1$, and $\mathcal{E}_2$.

Using $\cos^2(\vec{k}\cdot\vec{r}-\omega t) + \sin^2(\vec{k}\cdot\vec{r}-\omega t) = 1$ we obtain the equation for the electric field $\vec{\mathcal{E}}(\vec{r},t)$ at a given space time point

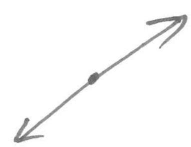
$$\sum_{i=1}^{2}\sum_{j=1}^{2} \mathcal{E}_i A_{ij} \mathcal{E}_j = 1$$

For homework, Find the 2×2 matrix A_{ij} and show that the electric field vector traces out an ellipse as time $\underline{t}$ elapses at a fixed space point $\vec{r}$. (Remember a straight line segment is a degenerate ellipse.

Because the tip of the electric field vector traces out an ellipse, we say that the generic electromagnetic wave is <u>elliptically polarized</u>.

E.g. (1) $\varphi_1 = 0$, $\varphi_2 = 0 \implies \vec{E}_0$ is a real vector

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$



<u>linear</u> polarization along $\vec{E}_0$.

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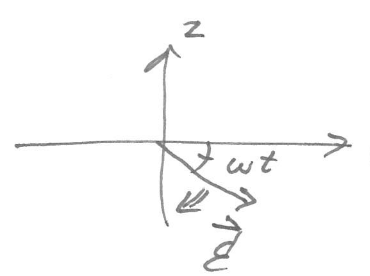
(2) $\varphi_1 = 0$, $\varphi_2 = -\frac{\pi}{2}$ and $|E_{01}| = \mathcal{E}_0 = |E_{02}|$

$$\mathcal{E}_1(\vec{r},t) = \mathcal{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$\mathcal{E}_2(\vec{r},t) = \mathcal{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$$

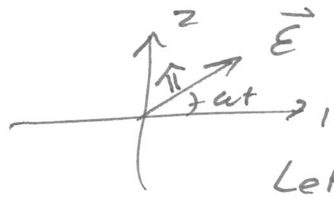


<u>Right Circular Polarization</u>

The wave comes out of the page $(\hat{k})$ and $\vec{\mathcal{E}}$ rotates <u>clockwise</u>. (I know, I know – it sounds backwards...)

③ $\varphi_1 = 0$, $\varphi_2 = +\frac{\pi}{2}$ and $|E_{01}| = \mathcal{E}_0 = |E_{02}|$

$$\mathcal{E}_1(\vec{r},t) = \mathcal{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$



Left Circular Polarization

$$\mathcal{E}_2(\vec{r},t) = -\mathcal{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$$

At $\vec{r}=0$ for example we find

$$\vec{\mathcal{E}}(r,t)\Big|_{\vec{r}=0} = \mathcal{E}_0\left[\vec{\epsilon}_1 \cos(\omega t) + \vec{\epsilon}_2 \sin(\omega t)\right]$$

The wave comes out of the page and $\vec{\mathcal{E}}$ rotates counterclockwise.

Poynting Vector

$$\vec{S}_{(\vec{r},t)} = \frac{1}{\mu_0}\vec{\mathcal{E}}_{(\vec{r},t)} \times \vec{\mathcal{B}}_{(\vec{r},t)}$$ ← all real (or $\vec{\mathcal{E}} \times \vec{\mathcal{H}}$)

write: $\vec{E}_0 = \vec{E}_{0R} + i\,\vec{E}_{0I}$

$$\vec{B}_{0R} = \frac{\vec{k}\times\vec{E}_{0R}}{\omega} \qquad \vec{B}_{0I} = \frac{\vec{k}\times\vec{E}_{0I}}{\omega}$$

Then: $\vec{\mathcal{E}}(\vec{r},t) = \vec{E}_{0R}\cos(\vec{k}\cdot\vec{r}-\omega t) - \vec{E}_{0I}\sin(\vec{k}\cdot\vec{r}-\omega t)$

$$\vec{\mathcal{B}}(\vec{r},t) = \frac{\vec{k}\times\vec{\mathcal{E}}}{\omega} = \frac{1}{v}\hat{k}\times\vec{\mathcal{E}}$$

$$\vec{\mathcal{S}}_{(\vec{r},t)} = \frac{1}{\mu_0}\vec{\mathcal{E}}\times\vec{\mathcal{B}} = \frac{1}{\mu_0 v}\vec{\mathcal{E}}\times(\hat{k}\times\vec{\mathcal{E}}) = \frac{1}{\mu_0 v}\hat{k}(\vec{\mathcal{E}}\cdot\vec{\mathcal{E}})$$

since $\hat{k}\cdot\vec{\mathcal{E}} = 0$

$$\vec{\mathcal{S}}(\vec{r},t) = \frac{\hat{k}}{\mu_0 v}\Big[\vec{E}_{0R}\cdot\vec{E}_{0R}\cos^2(\vec{k}\cdot\vec{r}-\omega t) + 2\vec{E}_{0R}\cdot\vec{E}_{0I}\cos(\vec{k}\cdot\vec{r}-\omega t)\sin(\vec{k}\cdot\vec{r}-\omega t) + \vec{E}_{0I}\cdot\vec{E}_{0I}\sin^2(\vec{k}\cdot\vec{r}-\omega t)\Big]$$

This real Poynting vector is time dependent, but if we average over one period $T = \frac{2\pi}{\omega}$ and use

$\langle\cos^2\rangle_{\tau} = \frac{1}{2} = \langle\sin^2\rangle$ and $\langle\sin\cos\rangle = 0$ we get

time avg

complex conjugate

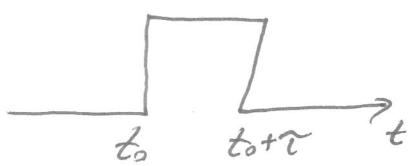
$$\langle\vec{\mathcal{S}}\rangle = \frac{1}{\mu_0 v}\frac{\hat{k}}{2}\left(E_{0R}^2 + E_{0I}^2\right) = \frac{\hat{k}}{2\mu_0 v}\vec{E}_0^*\cdot\vec{E}_0$$

Homework: Show that $\langle\vec{\mathcal{S}}\rangle \propto \langle u\rangle\hat{k}v$

where $\langle u\rangle$ is the electromagnetic energy density (energy / unit volume) averaged over one period. Find the proportionality constant and interpret the result physically.

$$\langle\vec{\mathcal{S}}\rangle = \frac{1}{2\mu}\vec{E}_0^*\cdot\vec{B}_0 = \frac{1}{2\mu}\vec{E}_0\cdot\vec{B}_0^*$$

Everything so far has been done for a plane wave. But generally we are looking at electromagnetic waves which are not of this nature. The usual source of radiation consists of many independent sources at the atomic level, each of which is in operation for a lifetime which is typically on the order of $\tau \sim 10^{-8}$ sec. Such a "wave train" contains $\frac{\omega\tau}{2\pi} \sim 10^6$ vibrations (for 500nm wavelength light). Such a finite train cannot be monochromatic since a wavetrain of duration τ must mix in a spread of frequencies $\Delta\omega \sim \frac{2\pi}{\tau}$ as can be seen by Fourier analyzing the finite pulse



or more simply by applying the uncertainty principle.

So the radiation is not strictly monochromatic. Consequently, the electric field vector will wander in a manner not determined by the plane wave analysis given above. Furthermore, each independent source has its own phase and the sources act independently and incoherently. Thus, the average of the Poynting vector over one period of a monochromatic plane wave $\langle \mathcal{S} \rangle$ must be replaced by an additional time average since the amplitude $\vec{E}_0$ is now slowly time varying. This additional time average should be made over a period that will be short compared to the resolution time of the detection apparatus used, but long compared to the period of the "monochromatic" wave. Thus we have

$$\langle \vec{\mathcal{L}} \rangle = \frac{1}{\mu v} \hat{k} \langle \vec{E}_0^*(t) \cdot \vec{E}_0(t) \rangle$$

where $\vec{E}_0(t)$ indicates that the electric field has a "slow" time variation (compared to $\frac{2\pi}{\omega}$ pseudo period)

Consider now the quantity

$$J_{ij} \equiv \langle E_{0i}(t)\, E_{0j}^*(t) \rangle = \vec{\epsilon}_i \cdot \langle \vec{E}_0(t)\, \vec{E}_0^*(t) \rangle \cdot \vec{\epsilon}_j$$

where $\underline{\underline{J}} = [J_{ij}]$ is a 2×2 matrix. This matrix contains <u>all</u> the information about the state of polarization of the radiation. Note that $\underline{\underline{J}}$ is hermitian since $J_{ij}^* = J_{ji}$ which is clear from its definition.

Also, we may write

$$\langle \vec{\mathcal{L}} \rangle = \frac{1}{2\mu v} \hat{k}\, Tr(\underline{\underline{J}})$$ where Tr is the <u>trace</u> operation

As we will see shortly, the matrix $\underline{\underline{J}}$ can be used to describe the <u>scattering</u> of radiation.

Actually, it is more useful to define another matrix

$$\underline{\underline{\rho}} \equiv \frac{\underline{\underline{J}}}{Tr(\underline{\underline{J}})}$$ so that $Tr(\underline{\underline{\rho}}) = 1$

This is called the <u>density matrix</u> for the radiation. Density matrices are of great use and importance for a wide variety of physical problems (elementary particle physics, nuclear physics, statistical mechanics, etc.) The present usage serves as an introduction.

In the case of a <u>monochromatic</u> plane wave, the time average shown in $\underline{\underline{J}}$ is irrelevant and we have

$$\underline{\underline{\rho}} = \frac{\underline{\underline{E_0}}\,\underline{\underline{E_0}}^\dagger}{\underline{\underline{E_0}}^\dagger\,\underline{\underline{E_0}}}$$ where $\underline{\underline{E_0}}$ is a complex 2-component <u>column</u> vector.

dagger $\dagger$ means "transpose complex conjugate" or hermitian conjugate

$\underline{\underline{E_0}}^\dagger$ is a 2-component <u>row</u> vector

$\underline{\underline{E_0}}\,\underline{\underline{E_0}}^\dagger \rightarrow 2\times 2$ matrix

$\underline{\underline{E_0}}^\dagger\,\underline{\underline{E_0}} \rightarrow 1\times 1$ scalar

We noted previously that a monochromatic plane wave is polarized (elliptically, in general). Thus polarized radiation is described by a density matrix with the property

$$\underline{\underline{\rho}}^2 = \underline{\underline{\rho}}\,\underline{\underline{\rho}} = \frac{\underline{\underline{E_0}}\,\underline{\underline{E_0}}^{+}\underline{\underline{E_0}}\,\underline{\underline{E_0}}^{+}}{(\underline{\underline{E_0}}^{+}\underline{\underline{E_0}})^2} = \frac{\underline{\underline{E_0}}\,\underline{\underline{E_0}}^{+}}{\underline{\underline{E_0}}^{+}\underline{\underline{E_0}}} = \underline{\underline{\rho}} \qquad \text{(Monochromatic)}$$

which in turn implies that

$$Tr(\underline{\underline{\rho}}^2) = Tr(\underline{\underline{\rho}}) = 1$$

So polarized wave $\Longrightarrow Tr(\underline{\underline{\rho}}^2) = 1$

The converse is also true: $Tr(\underline{\underline{\rho}}^2) = 1 \Rightarrow$ polarized but is not trivial to prove.

Proof: Note that the diagonal elements of $\underline{\underline{J}}$ are positive semidefinite

$$J_{ii} = \langle E_{0i} E_{0i}^{*} \rangle = \langle |E_{0i}|^2 \rangle \quad \text{thus } Tr(\underline{\underline{J}}) \geq 0$$

and hence $\rho_{ii} = \dfrac{J_{ii}}{Tr(\underline{\underline{J}})} \geq 0$

Since $Tr(\underline{\underline{\rho}}) = 1$ we must have $0 \leq \rho_{ii} \leq 1$