No matter how complicated the source of radiation, there are two independent beams of radiation each polarized elliptically and described by $\underline{\underline{u}}_{(1)}$ and $\underline{\underline{u}}_{(2)}$. For example, if $\epsilon = 0$ (no eccentricity) then the ellipses are circles and

$$\underline{\underline{u}}_{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\psi} \\ i e^{-i\psi} \end{pmatrix} = e^{-i\psi}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-i\psi}\frac{1}{\sqrt{2}}\left(\underline{\underline{\epsilon}}_{(1)} + i\,\underline{\underline{\epsilon}}_{(2)}\right)$$

$$\underline{\underline{u}}_{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} i e^{i\psi} \\ e^{i\psi} \end{pmatrix} = i e^{i\psi}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = i e^{i\psi}\frac{1}{\sqrt{2}}\left(\underline{\underline{\epsilon}}_{(1)} - i\,\underline{\underline{\epsilon}}_{(2)}\right)$$

The phase factors can be discarded and the remaining vectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ are polarization vectors for left and right circular polarization, respectively. This result follows because in ordinary vector notation

$$\vec{\mathcal{E}}_{\pm}(\vec{r},t) = \mathcal{E}_0 \, \mathrm{Re}\left[(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$= \mathcal{E}_0\left[\vec{\epsilon}_1 \cos(\vec{k}\cdot\vec{r}-\omega t) \mp \vec{\epsilon}_2 \sin(\vec{k}\cdot\vec{r}-\omega t)\right].$$

As before, we set $\vec{r}=0$ and find

$$\vec{\mathcal{E}}_{\pm}(0,t) = \mathcal{E}_0\left(\vec{\epsilon}_1 \cos\omega t \pm \vec{\epsilon}_2 \sin\omega t\right)$$

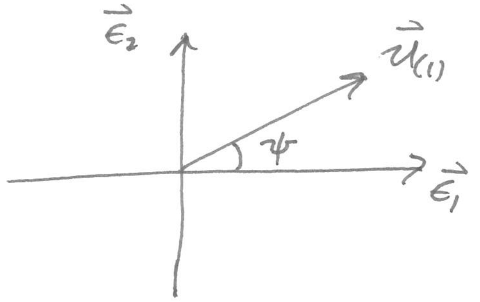
which identifies

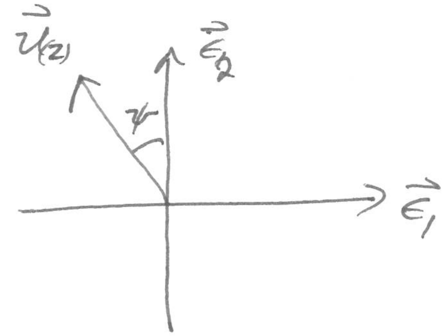
$$\text{LCP:} \quad \underline{\underline{\epsilon}}_L = \frac{1}{\sqrt{2}}\left(\underline{\underline{\epsilon}}_{(1)} + i\,\underline{\underline{\epsilon}}_{(2)}\right)$$

$$\text{RCP:} \quad \underline{\underline{\epsilon}}_R = \frac{1}{\sqrt{2}}\left(\underline{\underline{\epsilon}}_{(1)} - i\,\underline{\underline{\epsilon}}_{(2)}\right)$$

As a second example, we set $\epsilon = 1$ (the ellipses degenerate into straight lines) with

$$\underline{\underline{u}}_{(1)} = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \quad \text{and} \quad \underline{\underline{u}}_{(2)} = \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}$$





If $\rho_1 = \frac{1}{2} = \rho_2$ then $\underline{\underline{\rho}} = \frac{1}{2}\left[\underline{\underline{u}}_{(1)}\underline{\underline{u}}_{(1)}^{\dagger} + \underline{\underline{u}}_{(2)}\underline{\underline{u}}_{(2)}^{\dagger}\right] = \frac{1}{2}\mathbb{I}$

In this degenate case, any two orthonormal vectors perpendicular to the direction of propagation are eigenvectors. The radiation in this case is said to be <u>unpolarized</u>. Assuming $\rho_1 > \rho_2$ we define the polarization as

$$P \equiv \rho_1 - \rho_2 = 2\rho_1 - 1 \qquad (\text{since } \rho_1 + \rho_2 = 1).$$

For example, light from a lightbulb has $P = 0$ — the radiation is unpolarized. But there are many processes in which radiation can become polarized by scattering.

Finally, we note that the matrix $\underline{\underline{J}}$ (from which we defined the density matrix $\underline{\underline{\rho}}$) is connected to the so-called <u>Stokes parameters</u> S_0, S_1, S_2, S_3 in the following way:

$$S_0 = Tr(\underline{\underline{J}}) = J_{11} + J_{22} = \langle E_{01}(t) E_{01}^*(t) + E_{02}(t) E_{02}^*(t) \rangle$$

$$S_1 = J_{11} - J_{22} = \langle E_{01}(t) E_{01}^*(t) - E_{02}(t) E_{02}^*(t) \rangle$$

$$S_2 = 2 R_e (J_{21}) = J_{21} + J_{12} = \langle E_{02}(t) E_{01}^*(t) + E_{01}(t) E_{02}^*(t) \rangle$$

$$S_3 = 2 I_m (J_{21}) = \frac{J_{21} - J_{12}}{i} = \frac{1}{i} \langle E_{02}(t) E_{01}^*(t) - E_{01}(t) E_{02}^*(t) \rangle$$

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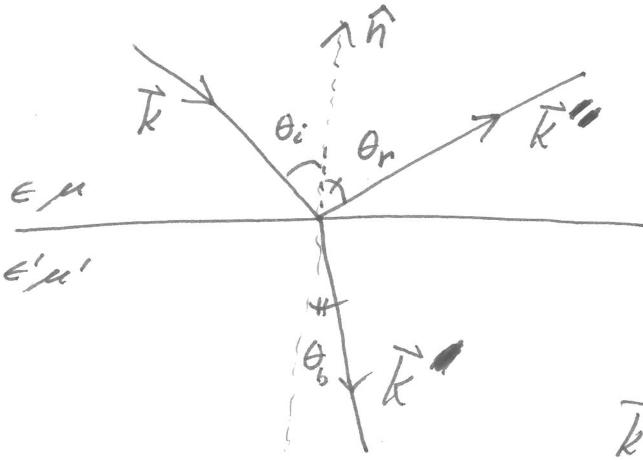
homework: Can $\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ be a density matrix?

For the density matrices $\underline{\underline{\rho}}_b = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\underline{\underline{\rho}}_c = \begin{pmatrix} \frac{1}{4} & \frac{i}{4} \\ -\frac{i}{4} & \frac{3}{4} \end{pmatrix}$

Find the eigenvalues and eigenvectors. Find the polarization P. What is the nature of the polarization? (linear, circular, etc.)?

Reflection and refraction from a dielectric interface

We now consider wave propagation in a medium which contains a planar discontinuity in the permittivity and permeability constants.



θ_i = angle of incidence

θ_r = angle of reflection

θ_b = angle of refraction (bending)

$\vec{k}$ = incident wave vector

$\vec{k}'$ = refracted wave vector

$\vec{k}''$ = reflected wave vector

incident wave: electric field $\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

refracted wave: $\vec{E}'(\vec{r},t) = \vec{E}_0'\, e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$

reflected wave: $\vec{E}''(\vec{r},t) = \vec{E}_0''\, e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$

The magnetic field associated with each of these waves is obtained from the general plane wave result

$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} \quad \text{so} \quad \underbrace{\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega}}_{\text{incident}}, \quad \underbrace{\vec{B}_0' = \frac{\vec{k}'\times\vec{E}_0'}{\omega}}_{\text{refracted}}, \quad \underbrace{\vec{B}_0'' = \frac{\vec{k}''\cdot\vec{E}_0''}{\omega}}_{\text{reflected}}$$

Note that all waves have the <u>same</u> frequency $\underline{\omega}$. Otherwise the boundary conditions at the interface can not be satisfied for all time.

incident: $\vec{B}(\vec{r},t) = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$
magnetic field

refracted: $\vec{B}'(\vec{r},t) = \vec{B}_0' \, e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$

reflected: $\vec{B}''(\vec{r},t) = \vec{B}_0'' \, e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$

Physically, the incident wave sets the molecules making up the dielectric media into motion which creates the reflected and refracted waves. The media are then forced to vibrate at the same driving frequency.

We assume that there are no <u>free</u> charges or currents on the interface. Then the boundary conditions are

$$\hat{n}\cdot(\vec{D}_R - \vec{D}_I) = 0 \qquad \hat{n}\times(\vec{E}_R - \vec{E}_I) = 0$$

$$\hat{n}\cdot(\vec{B}_R - \vec{B}_I) = 0 \qquad \hat{n}\times(\vec{H}_R - \vec{H}_I) = 0$$

where the subscipt <u>R</u> stands for the <u>refracted</u> medium and <u>I</u> stands for the <u>incident</u> medium. (The reflected wave is in the incident medium.)