Radiation

The potentials in Coulomb Gauge (also called Transverse Gauge); (also called Radiation Gauge):

In the previous analysis, we have been concerned with the propagation of plane waves in a variety of media both in the absence and presence of boundaries. We now look at how localized charges and currents give rise to electromagnetic waves. Here it is easiest to use potentials to determine the fields.

In the following, $\vec{E}$, $\vec{B}$, ρ, $\vec{J}$, Φ, and $\vec{A}$ are space- and time-dependent fields. Eg. $\Phi(\vec{r},t)$.

Maxwell Equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss)} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$$

(Ampère – Maxwell)

Continuity: $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

and the Coulomb Gauge condition is $\vec{\nabla}\cdot\vec{A} = 0$

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi - \frac{\partial}{\partial t}\cancel{(\vec{\nabla}\cdot\vec{A})}^{\,0} = \frac{\rho}{\epsilon_0} \implies \nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

same as in electro statics

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0$$

$$\vec{\nabla}\times\vec{E} = -\cancel{\vec{\nabla}\times(\vec{\nabla}\Phi)}^{\,0} - \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\cancel{(\vec{\nabla}\cdot\vec{A})}^{\,0} - \nabla^2\vec{A}$$

$$= -\nabla^2\vec{A} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left[-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right]$$

$$= \mu_0\vec{J} - \epsilon_0\mu_0\vec{\nabla}\frac{\partial\Phi}{\partial t} - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{A}$$

This last equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial\Phi}{\partial t} \equiv -\mu_0\vec{J}_{tr}$$

where $\vec{J}_{tr}(\vec{r},t)$ is the "transverse current."

According to the Helmholtz theorem, any vector field can be decomposed into a longitudinal (or irrotational) piece that has zero curl, and a transverse (or solenoidal) piece that has zero divergence.
$$\vec{J}(\vec{r},t) = \vec{J}_{long}(\vec{r},t) + \vec{J}_{tr}(\vec{r},t)$$

where $\vec{\nabla}\times\vec{J}_{long}(\vec{r},t)=0$ and $\vec{\nabla}\cdot\vec{J}_{tr}(\vec{r},t)=0$.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{J}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \nabla^2\vec{J}$$

$$\nabla^2\vec{J} = \vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{J})$$

$$\nabla^2\vec{J}_{tr} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{J})$$

$$\nabla^2\vec{J}_{long} = \vec{\nabla}(\vec{\nabla}\cdot\vec{J}) = -\vec{\nabla}\left(\frac{\partial\rho}{\partial t}\right)$$

We know how to solve the last equation:

$$\vec{J}_{long}(\vec{r},t) = \vec{\nabla}\frac{\partial}{\partial t}\iiint\frac{\rho(\vec{r}\,',t)\,dV'}{4\pi|\vec{r}-\vec{r}\,'|} = \epsilon_0\vec{\nabla}\frac{\partial\Phi(\vec{r},t)}{\partial t}$$

---

so on the bottom of the previous page

$$-\left(\mu_0\vec{J} - \epsilon_0\mu_0\vec{\nabla}\frac{\partial\Phi}{\partial t}\right) = -\mu_0(\vec{J}-\vec{J}_{long}) = -\mu_0\vec{J}_{tr}$$

To recapitulate:

$$\nabla^2 \Phi(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\epsilon_0} \qquad \text{same as in electro}\underline{\text{statics}}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = -\mu_0 \vec{J}_{tr}(\vec{r},t)$$

Now you can see why Coulomb Gauge is also called Transverse Gauge (the source of the vector potential is the transverse part of the current density) and Radiation Gauge (the vector potential is the solution to the non-homogeneous wave equation with the transverse current as its source). From Robert G. Brown's notes at Duke U., "only transverse currents give rise to purely transverse radiation fields far from the sources, with the static potential present but not giving rise to radiation."

The potentials can be found with two Green functions:

the Poisson Green Function $G_0(\vec{r}-\vec{r}\,') = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$

where $\nabla^2 G_0(\vec{r}-\vec{r}\,') = -4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')$

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\iiint G_0(\vec{r}-\vec{r}\,')\,\rho(\vec{r}\,',t)\,dV'$$

$$= \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\vec{r}\,',t)\,dV'}{|\vec{r}-\vec{r}\,'|}$$ same as electrostatics.

---

and the Green function for the wave operator

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}-\vec{r}\,'; t-t') = -4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

$$\vec{A}(\vec{r},t) = \vec{A}_0(\vec{r},t) + \frac{\mu_0}{4\pi}\int_{t'=-\infty}^{\infty} dt' \iiint dV'\, G(\vec{r}-\vec{r}\,'; t-t')\,\vec{J}_{tr}(\vec{r}\,',t')$$

where $\vec{A}_0(\vec{r},t)$ solves the homogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}_0(\vec{r},t) = 0$$ and also satisfies

the Coulomb Gauge condition $\vec{\nabla}\cdot\vec{A}_0(\vec{r},t) = 0$

and remember $\vec{\nabla}\cdot\vec{J}_{tr}(\vec{r},t) = 0$.

With this last equation, one can show that the vector potential $\vec{A}(\vec{r},t)$ satisfies the Coulomb gauge condition

$$\vec{\nabla}\cdot\vec{A}(\vec{r},t) = 0.$$

Now we go through the whole exercise again in Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\left(\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t}\right) = -\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

$$= -\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0 \quad \text{as in Coulomb Gauge}$$

$$\vec{\nabla}\times\vec{E} = -\cancelto{0}{\vec{\nabla}\times(\vec{\nabla}\Phi)} - \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} = -\frac{\partial \vec{B}}{\partial t} \quad \text{as in Coulomb Gauge}$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left[-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right]$$

$$\Rightarrow \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)}_{0} - \mu_0\vec{J} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = -\mu_0\vec{J}(\vec{r},t)$$

full current, not just the transverse part.

---

space and time are treated symmetrically

$\Box^2 A^\nu = -J^\nu$ with 4-vectors $x^\mu = (ct, \vec{r})$,

$\partial_\mu\partial^\mu A^\nu = -J^\nu$ $\quad A^\mu = \left(\frac{\Phi}{c}, \vec{A}\right)$, $J^\mu = \left(\frac{\rho}{c}, \vec{J}\right)$.

The potentials are obtained from the sources as:

$$\Phi(\vec{r},t) = \Phi_0(\vec{r},t) + \frac{1}{4\pi\epsilon_0}\int_{t'=-\infty}^{\infty} dt' \iiint dV'\, G(\vec{r}-\vec{r}\,';t-t')\, \rho(\vec{r}\,',t')$$

$$\vec{A}(\vec{r},t) = \vec{A}_0(\vec{r},t) + \frac{\mu_0}{4\pi}\int_{t'=-\infty}^{\infty} dt' \iiint dV'\, G(\vec{r}-\vec{r}\,';t-t')\, \vec{J}(\vec{r}\,',t')$$

where
$$\left.\begin{aligned} &\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi_0(\vec{r},t) = 0 \\ &\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}_0(\vec{r},t) = 0 \end{aligned}\right\} \text{homogeneous wave equations}$$

and
$$\vec{\nabla}\cdot\vec{A}_0(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_0(\vec{r},t) = 0$$

With the continuity equation $\vec{\nabla}\cdot\vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\rho(\vec{r},t) = 0$ one can demonstrate that the potentials satisfy the Lorentz gauge condition

$$\vec{\nabla}\cdot\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi(\vec{r},t) = 0$$

In the following, we introduce time Fourier transforms

$$\mathcal{X}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\infty} d\omega \, \tilde{\mathcal{X}}_\omega(\vec{r}) \, e^{-i\omega t}$$

(note that $\tilde{\mathcal{X}}_\omega(\vec{r}) = \tilde{\mathcal{X}}^*_{-\omega}(\vec{r})$ if $\mathcal{X}(\vec{r},t)$ is <u>real</u>.)

and a further space Fourier transform

$$\mathcal{X}(\vec{r},t) = \int_{\omega=-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \iiint \frac{d^3k}{(\sqrt{2\pi})^3} \, \tilde{\tilde{\mathcal{X}}}_{\vec{k}\omega} \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\mathcal{X}(\vec{r},t)$ is any space time dependent field.

The transverse current is easily defined in the space Fourier transformed field

$$\left(\tilde{\vec{J}}_{\vec{k}}\right)_{tr} = \tilde{\vec{J}}_{\vec{k}} - \frac{\vec{k}(\vec{k}\cdot\tilde{\vec{J}}_{\vec{k}})}{k^2}$$

and therefore $\vec{k}\cdot\left(\tilde{\vec{J}}_{\vec{k}}\right)_{tr} = 0$

We will also need the time Fourier transform

$$\left(\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)_{tr} = \tilde{\tilde{\vec{J}}}_{\vec{k}\omega} - \frac{\vec{k}(\vec{k}\cdot\tilde{\tilde{\vec{J}}}_{\vec{k}\omega})}{k^2}$$

with $\vec{k}\cdot\left(\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)_{tr} = 0$

Under a time Fourier transform, the inhomogeneous wave equation for $G(\vec{r}-\vec{r}\,'; t-t')$ becomes

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{G}_\omega(\vec{r}-\vec{r}\,') = -4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

This is now the inhomogeneous Helmholtz equation.

Note that for $\omega \to 0$ we recover the Poisson Green function $G_0(\vec{r}-\vec{r}\,') = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$.

The singularity in $\tilde{G}_\omega(\vec{r}-\vec{r}\,')$ all comes from the $\dfrac{1}{|\vec{r}-\vec{r}\,'|}$, so we write

$$\tilde{G}_\omega(\vec{r}-\vec{r}\,') = \frac{f_\omega(|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}$$

where we used the isotropy of space to have f_ω depend only on $|\vec{r}-\vec{r}\,'| \equiv R$. Then for $R \neq 0$ (that is, for $\vec{r} \neq \vec{r}\,'$) we have

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{G}_\omega(\vec{r}-\vec{r}\,') = \left(\frac{1}{R^2}\frac{\partial}{\partial R} R^2 \frac{\partial}{\partial R} + \frac{\omega^2}{c^2}\right) \frac{f_\omega(R)}{R} = 0$$

or

$$\frac{d^2 f(R)}{dR^2} + \frac{\omega^2}{c^2} f(R) = 0$$

Several choices are:

$$f_\omega^r(R) = e^{i\frac{\omega}{c}R}, \quad f_\omega^a(R) = e^{-i\frac{\omega}{c}R}, \quad f_\omega^s(R) = \cos\left(\frac{\omega}{c}R\right)$$

for which

$$\tilde{G}_\omega^r(R) = \frac{e^{i\frac{\omega}{c}R}}{R}$$ the <u>retarded</u> Green function

$$\tilde{G}_\omega^a(R) = \frac{e^{-i\frac{\omega}{c}R}}{R}$$ the <u>advanced</u> Green function

$$\tilde{G}_\omega^s(R) = \frac{\cos\left(\frac{\omega}{c}R\right)}{R}$$ the <u>standing</u> Green function

The behavior at $R=0$ (that is $\vec{r}=\vec{r}\,'$) is guaranteed by the $\frac{1}{R}$ — it determines the delta function behavior. Note that $f_\omega(R) = \sin\left(\frac{\omega}{c}R\right)$ is <u>not</u> permitted since $\frac{\sin\left(\frac{\omega}{c}R\right)}{R} \xrightarrow[R\to 0]{} \frac{\omega}{c}$ which is <u>not</u> singular.