3. (a) Show that the column vector $\vec{v}_{(1)}$ (defined in lecture) describes an ellipse with major axis along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and minor axis along $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eccentricity e. Show that the column vector $\vec{v}_{(2)}$ describes an ellipse with major axis along $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and minor axis along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eccentricity e.

$\vec{v}_{(1)}$ and $\vec{v}_{(2)}$ are polarization vectors

$$\vec{v}_{(1)} = \frac{1}{\sqrt{2-\epsilon^2}}\begin{pmatrix} 1 \\ i\sqrt{1-\epsilon^2} \end{pmatrix} \qquad \vec{v}_{(2)} = \frac{1}{\sqrt{2-\epsilon^2}}\begin{pmatrix} i\sqrt{1-\epsilon^2} \\ 1 \end{pmatrix}$$

First consider light polarized along $\vec{v}_1$ only with <u>real</u> amplitude $\mathcal{E}_{01}$

$$\vec{E}(\vec{r},t) = \mathcal{E}_{01}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\vec{v}_{(1)}$$

$$= Re\left[\vec{E}(\vec{r},t)\right] + i\, Im\left[\vec{E}(\vec{r},t)\right]$$

$$= \vec{A} + i\vec{B} \qquad \text{with } \vec{A}, \vec{B} \text{ real.}$$

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \frac{\mathcal{E}_{01}}{\sqrt{2-\epsilon^2}}\cos(\vec{k}\cdot\vec{r}-\omega t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{\mathcal{E}_{01}\sqrt{1-\epsilon^2}}{\sqrt{2-\epsilon^2}}\sin(\vec{k}\cdot\vec{r}-\omega t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \frac{\mathcal{E}_{01}}{\sqrt{2-\epsilon^2}}\sin(\vec{k}\cdot\vec{r}-\omega t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\mathcal{E}_{01}\sqrt{1-\epsilon^2}}{\sqrt{2-\epsilon^2}}\cos(\vec{k}\cdot\vec{r}-\omega t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eliminate the phase angle $(\vec{k}\cdot\vec{r}-\omega t)$ as in problem 1. From either of these equations (they are redundant) we obtain:

$$\frac{A_x^2}{a_1^2} + \frac{A_y^2}{a_2^2} = 1 = \frac{B_x^2}{a_1^2} + \frac{B_y^2}{a_2^2}$$

where $a_1 = \frac{\mathcal{E}_{01}}{\sqrt{2-\epsilon^2}}$ and $a_2 = \frac{\mathcal{E}_{01}\sqrt{1-\epsilon^2}}{\sqrt{2-\epsilon^2}}$

$a_1 \geq a_2$ as ϵ ranges from 0 to 1 with equality only at $\epsilon = 0$.

Thus $\vec{v}_{(1)}$ describes an ellipse with major axis along $\binom{1}{0}$ and minor axis along $\binom{0}{1}$.

ϵ is the eccentricity of the ellipse by definition:

$$\frac{a_2}{a_1} = \sqrt{1-\epsilon^2}$$

[Aside: $\sqrt{1-\epsilon^2}$ is called the "ellipticity."]

A similar analysis holds for $\vec{v}_{(2)}$, but

$$\frac{a_1}{a_2} = \sqrt{1-\epsilon^2}$$

so $\vec{v}_{(2)}$ describes an ellipse with major axis along $\binom{0}{1}$ and minor axis along $\binom{1}{0}$. ∎

(b) Show that $\vec{v}^{\dagger}_{(i)} \cdot \vec{v}_{(j)} = \delta_{ij}$.

(c) Show that $\mathbf{R}^{\dagger}\mathbf{R} = \mathbf{I}$ and hence that $\vec{u}^{\dagger}_{(i)} \cdot \vec{u}_{(j)} = \delta_{ij}$.

(d) What geometric figure do the vectors $\vec{u}_{(i)}$ describe?

(e) Write the density matrix ρ as a single matrix.

b) $$\vec{v}_{(1)}^{\dagger} \cdot \vec{v}_{(2)} = \frac{1}{2-\epsilon^2}\left(1, -i\sqrt{1-\epsilon^2}\right)\begin{pmatrix} i\sqrt{1-\epsilon^2} \\ 1 \end{pmatrix} = 0$$

$$\vec{v}_{(1)}^{\dagger} \cdot \vec{v}_{(1)} = \frac{1}{2-\epsilon^2}\left(1, -i\sqrt{1-\epsilon^2}\right)\begin{pmatrix} 1 \\ i\sqrt{1-\epsilon^2} \end{pmatrix} = \frac{1}{2-\epsilon^2}\left[1 + (1-\epsilon^2)\right] = 1$$

$$\vec{v}_{(2)}^{\dagger} \cdot \vec{v}_{(2)} = \frac{1}{2-\epsilon^2}\left(-i\sqrt{1-\epsilon^2}, 1\right)\begin{pmatrix} i\sqrt{1-\epsilon^2} \\ 1 \end{pmatrix} = 1 \quad \blacksquare$$

---

c) $$R = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$ R is real

$$R^{\dagger}R = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

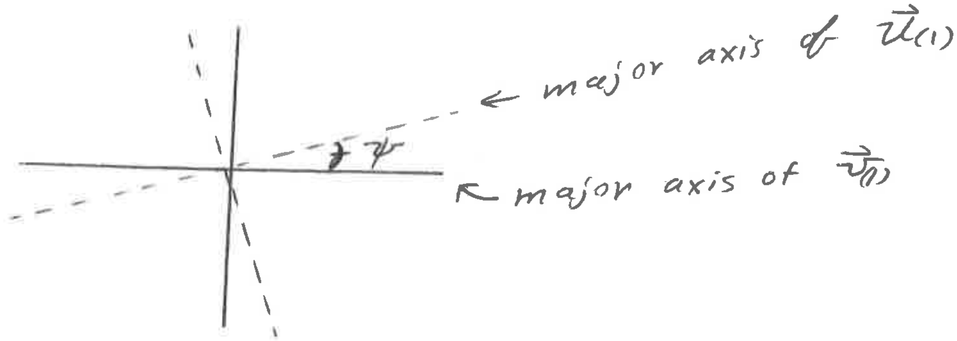
$$= \begin{pmatrix} \cos^2\psi + \sin^2\psi & 0 \\ 0 & \sin^2\psi + \cos^2\psi \end{pmatrix} = \mathbb{I}_2$$

$$\vec{u}_{(i)} = R\,\vec{v}_{(i)} \qquad \vec{u}^{\dagger}_{(i)} = \vec{v}^{\dagger}_{(i)} R^{\dagger}$$

$$\vec{u}^{\dagger}_{(i)} \cdot \vec{u}_{(j)} = \vec{v}^{\dagger}_{(i)} R^{\dagger} R\,\vec{v}_{(j)} = \vec{v}^{\dagger}_{(i)} \cdot \vec{v}_{(j)} = \delta_{ij}$$

$\blacksquare$

d) R rotates the principal axes of the ellipses described by $\vec{v}_{(1)}$ and $\vec{v}_{(2)}$ from the coordinate axes by the angle ψ counterclockwise



e) $$\rho = \rho_1 \vec{u}_{(1)} \vec{u}_{(1)}^{\dagger} + \rho_2 \vec{u}_{(2)} \vec{u}_{(2)}^{\dagger}$$

where $\rho_1 + \rho_2 = 1$ and $\rho_1 - \rho_2 \equiv P$

so in terms of the polarization, P:

$$\rho_1 = \frac{P+1}{2} \qquad \rho_2 = \frac{1-P}{2}$$

$$\rho = R\left[\frac{1+P}{2} \vec{v}_{(1)} \cdot \vec{v}_{(1)}^{\dagger} + \frac{1-P}{2} \vec{v}_{(2)} \cdot \vec{v}_{(2)}^{\dagger}\right] R^{\dagger}$$

$$= \frac{1}{2} \mathbb{I}_2 + \frac{P}{2} R\left[\vec{v}_{(1)} \vec{v}_{(1)}^{\dagger} - \vec{v}_{(2)} \vec{v}_{(2)}^{\dagger}\right] R^{\dagger}$$

where we used $\vec{v}_{(1)} \vec{v}_{(1)}^{\dagger} + \vec{v}_{(2)} \vec{v}_{(2)}^{\dagger} = \mathbb{I}_2$ (completeness)

$$\vec{v}_{(1)}\vec{v}_{(1)}^{\dagger} = \frac{1}{2-\epsilon^2}\begin{pmatrix} 1 \\ i\sqrt{1-\epsilon^2} \end{pmatrix}\left(1, -i\sqrt{1-\epsilon^2}\right) = \frac{1}{2-\epsilon^2}\begin{pmatrix} 1 & -i\sqrt{1-\epsilon^2} \\ i\sqrt{1-\epsilon^2} & 1-\epsilon^2 \end{pmatrix}$$

$$\vec{v}_{(2)}\vec{v}_{(2)}^{\dagger} = \frac{1}{2-\epsilon^2}\begin{pmatrix} 1-\epsilon^2 & i\sqrt{1-\epsilon^2} \\ -i\sqrt{1-\epsilon^2} & 1 \end{pmatrix}$$

$$\frac{1}{2}\left[\vec{v}_{(1)}\vec{v}_{(1)}^{\dagger} - \vec{v}_{(2)}\vec{v}_{(2)}^{\dagger}\right] = \frac{1}{2-\epsilon^2}\begin{pmatrix} \frac{\epsilon^2}{2} & -i\sqrt{1-\epsilon^2} \\ i\sqrt{1-\epsilon^2} & -\epsilon^2/2 \end{pmatrix}$$

$$\frac{1}{2}R\left[\vec{v}_{(1)}\vec{v}_{(1)}^{\dagger} - \vec{v}_{(2)}\vec{v}_{(2)}^{\dagger}\right]$$

$$= \frac{1}{2-\epsilon^2}\begin{pmatrix} \frac{\epsilon^2}{2}\cos\psi - i\sqrt{1-\epsilon^2}\sin\psi & -i\sqrt{1-\epsilon^2}\cos\psi + \frac{\epsilon^2}{2}\sin\psi \\ \frac{\epsilon^2}{2}\sin\psi + i\sqrt{1-\epsilon^2}\cos\psi & -\frac{\epsilon^2}{2}\cos\psi - i\sqrt{1-\epsilon^2}\sin\psi \end{pmatrix}$$

So finally

$$\rho = \frac{1}{2}\mathbb{I}_2 + \frac{P}{2-\epsilon^2}\begin{pmatrix} \frac{\epsilon^2}{2}\cos(2\psi) & \frac{\epsilon^2}{2}\sin(2\psi) - i\sqrt{1-\epsilon^2} \\ \frac{\epsilon^2}{2}\sin(2\psi) + i\sqrt{1-\epsilon^2} & -\frac{\epsilon^2}{2}\cos(2\psi) \end{pmatrix}$$

Thus a <u>real</u> density matrix implies $\epsilon = 1$, that is, <u>linear</u> polarization.

$\blacksquare$