Differential Forms and
Exterior Derivatives

Part I: Differential Forms

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Well do I remember being an undergraduate physics major when the legendary general relativity book *Gravitation* by Charles Misner, Kip Thorne, and John Wheeler was published.[1] We organized a seminar to learn as much as we could from *Gravitation* in the time we had. The first chapter, "Geometrodynamics in Brief" with its celebrated Parable of the Apple, was superbly enlightening. But the subsequent chapters that reviewed special relativity, which we had already studied elsewhere, were met with bewilderment by our little study group. The epigram welcoming us into these sections read, "Physics in Flat Spacetime: Wherein the reader meets an old friend, Special Relativity, outfitted in a new, mod attire, and becomes more intimately acquainted with her charms."[2]

Among other delights, the "new, mod attire" reaches its full glory with the mathematics of differential forms and exterior derivatives.[3] Climbing that learning curve was a challenge we met poorly at that time because we could not appreciate the motivation behind these strange new tools. Of these tools, Steven Weinberg wrote,

> Antisymmetric tensors and their antisymmetrized derivatives possess certain remarkably simple and useful properties. . . . In order to deal with these properties in a unified way, mathematicians have developed a general formalism, known as the *theory of differential forms*. Unfortunately, the rather abstract and compact notation associated with this formalism has in recent years seriously impeded communication between mathematicians and physicists.[4]

Let me suggest here a motivation that, had we seen it back then, would have inspired us undergraduate physicists to joyously engage with differential forms and exterior derivatives.

In previous courses we had encountered the vector identities

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = \mathbf{0} \tag{1}$$

and

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0, \tag{2}$$

which hold for any scalar field ϕ and any vector field $\mathbf{A}$. Had someone mentioned at the beginning of our seminar that these two identities are special cases of one bizarre equation,

$$\partial \Lambda (\partial \Lambda \psi) = 0, \tag{3}$$

we would have had reasons from within to dive into this mathematics that was new to us. Equation (3), known as Poincaré's lemma, says the second "exterior derivative" of ψ always vanishes, where ψ is something called a "differential form." If ψ is a differential 0-form, then Eq. (3) becomes Eq. (1), and if ψ is a differential 1-form, then Eq. (3) becomes Eq. (2)! To engage these topics, first we must define the "exterior product" of vectors, or "*r*-forms" in *N*-dimensional space, where *r* = 1, 2, . . . , *N*; second, extend *r*-forms to differential forms; and third, combine differential forms with exterior products to invent the exterior derivative. Then we will demonstrate examples of the exterior derivative's utility with electrodynamics by rewriting Maxwell's equations in this "new, mod attire."

Since integration forms the inverse operation of differentiation, the language of differential forms also unifies Gauss's divergence theorem,

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{E})\, dV \;\; = \oint_S (\mathbf{E} \cdot \hat{\mathbf{n}}) dS, \tag{4}$$

with Stoke's theorem,

$$\int_{S'} (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \hat{\mathbf{n}}\, dS = \oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} \tag{5}$$

(where volume V is bounded by closed surface S, and closed contour C forms the boundary of nonclosed surface S'). This article, Part I of a two-part series, introduces the exterior product and differential forms. Part II takes us into exterior derivatives and integrals of differential forms, and the Maxwell's equations example.

Exterior Product

The exterior product extends the antisymmetric cross product $\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$ beyond three-dimensional Euclidean geometry. Recall that $|\mathbf{A}\times\mathbf{B}|$ equals the area of the parallelogram whose sides are $\mathbf{A}$ and $\mathbf{B}$, and to orient $\mathbf{A}\times\mathbf{B}$ requires embedding $\mathbf{A}$ and $\mathbf{B}$ in three dimensions through the right-hand rule with its normal vector $\hat{\mathbf{n}}$. The exterior product, also called the wedge product or bivector, denoted $\vec{A}\wedge\vec{B}$, requires no higher-dimension embedding space. Note the notation $\vec{A}$ and $\vec{B}$ for vectors in arbitrary spaces; boldface is here reserved for vectors in Euclidean 3-space.

Like the cross product, the magnitude of the wedge product equals the area of a parallelogram whose sides are $\vec{A}$ and $\vec{B}$. In Euclidean space, a bivector's direction is equivalent to the right-hand rule, but $\hat{\mathbf{n}}$ is not essential to the bivector's definition. By analogy, the surface of a dining-room table can be given a sign without $\hat{\mathbf{n}}$: agree that passing food counterclockwise assigns the table a positive sign, and passing clockwise a minus sign. By definition the exterior product is antisymmetric,

$$\vec{A}\wedge\vec{B}=-\vec{B}\wedge\vec{A}, \tag{6}$$

and therefore for any $\vec{A}$,

$$\vec{A}\wedge\vec{A}=0. \tag{7}$$

The exterior product is associative, $\vec{A}\wedge(\vec{B}\wedge\vec{C})=\overrightarrow{(A\wedge B)}\wedge\vec{C}$, and distributes over addition, $(\vec{A}+\vec{B})\wedge\vec{C}=(\vec{A}\wedge\vec{C})+(\vec{B}\wedge\vec{C})$.

Let vectors be expanded over a basis $\vec{A}=A^{\mu}\vec{e}_{\mu}$ (repeated upper and lower indices are summed unless stated otherwise; $\vec{e}_{\mu}$ is a basis vector but, in general, not necessarily a unit vector). The exterior product becomes $\vec{A}\wedge\vec{B}=A^{\mu}B^{\nu}(\vec{e}_{\mu}\wedge\vec{e}_{\nu})$.

Let's work an explicit example. Suppose $\vec{A}=a\vec{e}_1+b\vec{e}_2+c\vec{e}_3$ and $\vec{B}=u\vec{e}_1+v\vec{e}_2+w\vec{e}_3$. Then

$$\vec{A}\wedge\vec{B}=(a\vec{e}_1+b\vec{e}_2+c\vec{e}_3)\wedge(u\vec{e}_1+v\vec{e}_2+w\vec{e}_3). \tag{8}$$

Using wedge product properties, this becomes

$$\vec{A}\wedge\vec{B}=(av-bu)(\vec{e}_1\wedge\vec{e}_2)+(cu-aw)(\vec{e}_3\wedge\vec{e}_1)$$
$$+(bw-cv)(\vec{e}_2\wedge\vec{e}_3). \tag{9}$$

Let cyclic ordering of (1, 2, 3) be used to denote $C^{12}\equiv C^3$ as the coefficient of $(\vec{e}_1\wedge\vec{e}_2)$, $C^{31}\equiv C^2$ as the coefficient of $(\vec{e}_3\wedge\vec{e}_1)$, and $C^{23}\equiv C^1$ as the coefficient of $(\vec{e}_2\wedge\vec{e}_3)$. Then, by requiring $C^{ij}=-C^{ji}$, Eq. (9) can be written in three equivalent ways:

$$\begin{aligned}\vec{A}\wedge\vec{B} &= \sum_{ijk\ cyclic} C^{i}\,(\vec{e}_j\wedge\vec{e}_k)\\ &= \sum_{i<j} C^{ij}(\vec{e}_i\wedge\vec{e}_j)\\ &= \frac{1}{2}\sum_{i,j} C^{ij}(\vec{e}_i\wedge\vec{e}_j).\end{aligned} \tag{10}$$

These procedures extend to non-Euclidean spaces and to more than three dimensions.

With the exterior product, in a space of N dimensions an r-vector is defined according to

$$\sum_{\{\mu\}} C^{\mu_1\mu_2\cdots\mu_r}(\vec{e}_{\mu_1}\wedge\vec{e}_{\mu_2}\wedge\cdots\wedge\vec{e}_{\mu_r}), \tag{11}$$

where $\mu_1<\mu_2<\cdots<\mu_r$ and $r\leq N$. The distinction between the 3-form $(d\vec{x}\wedge d\vec{y}\wedge d\vec{z})$ and the usual volume element $dxdydz$ is that the former is a *directed* volume element that distinguishes left- and right-handed coordinate systems.

Differential Forms

The move from r-vectors to differential forms proceeds by rescaling basis vectors with the corresponding coordinate differential: $\vec{e}_{\mu}\to d\vec{x}^{\mu}\equiv dx^{\mu}\vec{e}_{\mu}$ (no sum over μ). An elementary example would be $dx\,\hat{\mathbf{i}}\equiv d\mathbf{x}$. A differential 0-form, having no vector, is a scalar. A differential 1-form (call it $\tilde{\omega}$) has this structure (sum over μ):

$$\tilde{\omega}=A_{\mu}d\vec{x}^{\mu}. \tag{12}$$

This looks weird: one wants to denote the left-hand side of Eq. (12) as $d\tilde{\omega}$ because of the differentials $d\vec{x}^{\mu}$ on the right-hand side. But the notation in Eq. (12) is standard, and, as Charles Dickens wrote in another context, "The wisdom of our ancestors is in the simile; and my unhallowed hands shall not disturb it, or the Country's done for." [5] Justification for this unsettling notation emerges in Part II when we get to integrals of differential forms.

A differential 2-form $\tilde{\tilde{\rho}}$ is defined as

$$\tilde{\tilde{\rho}}=\sum_{\mu,\nu,\xi\ cyclic} C^{\mu}(d\vec{x}^{\nu}\wedge d\vec{x}^{\xi}), \tag{13}$$

where the C^{μ} can be renamed $C^{\nu\xi}=-C^{\xi\nu}$ with cyclic (μ,ν,ξ). A three-dimensional Euclidean example of Eq. (13) would be $C^{x}(d\vec{y}\wedge d\vec{z})+C^{y}(d\vec{z}\wedge d\vec{x})+C^{z}(d\vec{x}\wedge d\vec{y})$, which is the 2-form version of $\mathbf{C}=C_x\hat{\mathbf{i}}+C_y\hat{\mathbf{j}}+C_z\hat{\mathbf{k}}$.

We are now equipped to introduce the exterior derivative and show the claimed unification of identities in the differential and integral calculus of vectors. That project will occupy us in Part II, to be published in the Spring 2025 issue of *Radiations*.

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References

1. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (W. H. Freeman & Co., 1973).
2. Ibid., p. 45.
3. Ibid., pp. 91-98; Harley Flanders, *Differential Forms with Applications to the Physical Sciences* (Academic Press, 1963); and Dwight E. Neuenschwander, *Tensor Calculus for Physics* (Johns Hopkins Univ. Press, 2015), Ch. 8. These founders of the exterior calculus were Hermann Grassmann (1809-1877) and William Clifford (1845-1879). Clifford extended these ideas to "geometric algebra, "where the vector product denoted by juxtaposition $\vec{a}\vec{b} \equiv (\vec{a} \cdot \vec{b}) + (\vec{a} \wedge \vec{b})$ is developed in N dimensions. For a brief description of geometric algebra, see David Hestenes, Space-Time Algebra (Gordon & Breach, 1966).
4. Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, 1972), pp. 113-114.
5. From the opening lines of *A Christmas Carol* by Charles Dickens (1843).