Differential Forms and
Exterior Derivatives

Part II: Exterior Derivatives

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The Exterior Derivative

In Part I of this two-part series, published in the Fall 2024 issue of *Radiations*, we met differential forms and exterior products.[1] Recall that in a space of N dimensions the differential form "r-vector" is defined according to

$$\sum_{\mu} C^{\mu_1\mu_2\cdots\mu_r}\left(\vec{e}_{\mu_1}\wedge\vec{e}_{\mu_2}\wedge\cdots\wedge\vec{e}_{\mu_r}\right), \tag{1}$$

where $\mu_1 < \mu_2 < \cdots < \mu_r$ and $r \le N$, and the antisymmetric wedge product of two vectors makes a directed area. The differential form (illustrated here with a 2-form) has the structure

$$\sum_{\mu,\nu,\xi\ cyclic} C^{\mu}(d\vec{x}^{\nu}\wedge d\vec{x}^{\xi}), \tag{2}$$

where the $d\vec{x}^{\mu}$ are basis vectors rescaled by the corresponding coordinate differential. Thus equipped with differential forms, the exterior derivative now falls readily to hand. The exterior derivative of an r-form $\tilde{\tilde{T}}$, tildes optional, produces a differential (r+1)-form $\partial\wedge T$, alternatively denoted $\mathbf{d}T$, defined as the sum over μ of

$$\partial\wedge T \ \equiv\ d\vec{x}^{\mu}\wedge\left(\partial_{\mu}T\right). \tag{3}$$

(In applications to special and general relativity, $d\vec{t} = d\vec{x}^{0}$ must be introduced for the time dimension in spacetime.) If T is a scalar function—a 0-form ϕ—then Eq. (3) gives a 1-form—the vector $(\partial_{\mu}\phi)\ d\vec{x}^{\mu}$. (For scalar ϕ there is no vector for the wedge to "hook on to," so the wedge is as benign as multiplying by 1.) In Euclidean 3-space in rectangular coordinates, this becomes

$$\partial\wedge\phi\ (\partial_x\phi)dx\,\hat{\mathbf{i}} + \left(\partial_y\phi\right)dy\,\hat{\mathbf{j}} + (\partial_z\phi)dz\,\hat{\mathbf{k}}. \tag{4}$$

Now let T be the differential 1-form, $\tilde{\omega} = A_x d\vec{x} + A_y d\vec{y} + A_z d\vec{z} = A_j d\vec{x}^{\,j}$, and evaluate its exterior derivative. Equation (3) gives

$$\partial\wedge\tilde{\omega} = \left(\partial_i A_j\right)\left(d\vec{x}^{\,i}\wedge d\vec{x}^{\,j}\right). \tag{5}$$

In Cartesian coordinates the sum over *x,y,z* and wedge product antisymmetry turns Eq. (5) into

$$\partial\wedge\tilde{\omega} = \left(\partial_y A_z - \partial_z A_y\right)(d\vec{y}\wedge d\vec{z}) + (\partial_z A_x - \partial_x A_z)(d\vec{z}\wedge d\vec{x}) + \left(\partial_x A_y - \partial_y A_x\right)(d\vec{x}\wedge d\vec{y}). \tag{6}$$

These coefficients of the wedge products are, in three dimensions, the components of $\nabla\times\mathbf{A}$. Unlike the cross-product, the exterior product generalizes to higher dimensions. The reader may show that the exterior derivative of a 2-form in three-dimensional Euclidean space contains the familiar Euclidean divergence, because when $\tilde{\tilde{T}}$ is a 2-form like Eq. (5), then

$$\partial\wedge\tilde{\tilde{T}} = (\nabla\cdot\mathbf{A})(d\vec{x}\wedge d\vec{y}\wedge d\vec{z}). \tag{7}$$

We are now in a position to state and prove Poincaré's lemma,

$$\partial\wedge(\partial\wedge T) = 0, \tag{8}$$

where an arbitrary r-form T may in general be written in a notation simplified from Eq. (1),

$$T = \sum_{R} C_R\ (d\vec{x}^{1}\wedge d\vec{x}^{2}\wedge\cdots\wedge d\vec{x}^{r}), \tag{9}$$

where R denotes terms in the sum over all permutations of the wedge products consistent with their antisymmetry. The first exterior derivative of T is

$$\partial\wedge T = \sum_{R}\frac{\partial C_R}{\partial x^{\nu}}\ (dx^{\nu}\wedge d\vec{x}^{1}\wedge d\vec{x}^{2}\wedge\cdots\wedge d\vec{x}^{r}), \tag{10}$$

and for the second exterior derivative we find

$$\partial\wedge(\partial\wedge T) = \sum_R \frac{\partial^2 C_R}{\partial x^\mu \partial x^\nu} \,(d\vec{x}^\mu \wedge d\vec{x}^\nu \wedge d\vec{x}^1 \wedge d\vec{x}^2 \wedge \cdots \wedge d\vec{x}^r). \quad (11)$$

Now comes the punch line: Since μ and ν are dummy variables that are summed out, we may interchange them. But

$$\frac{\partial^2 C_R}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 C_R}{\partial x^\nu \partial x^\mu}, \qquad \text{while } d\vec{x}^\mu \wedge dx^\nu = -d\vec{x}^\nu \wedge dx^\mu. \quad (12)$$

Therefore, $\partial\wedge(\partial\wedge T) = -\partial\wedge(\partial\wedge T) = 0$.

Now we are equipped to unify $\nabla \times (\nabla\phi) = \mathbf{0}$ and $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. Suppose T is a 0-form, some scalar function ϕ. Then Poincaré's lemma in Cartesian coordinates becomes

$$\partial\wedge(\partial\wedge\phi) = \partial\wedge\left(d\vec{x}^j \frac{\partial\phi}{\partial x^j}\right) = \left(d\vec{x}^i \wedge d\vec{x}^j\right)\frac{\partial^2\phi}{\partial x^i \partial x^j}. \quad (13)$$

This clearly vanishes because, again, when exchanging i and j the second derivative is symmetric but the wedge product antisymmetric. So it remains to show that Eq. (13) contains $\nabla \times (\nabla\phi)$ in Euclidean 3-space. In familiar notation, but temporarily using subscripts to denote the partial derivatives of ϕ, the curl of a gradient may be calculated according to

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\jmath} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ \phi_x & \phi_y & \phi_z \end{vmatrix}$$

$$= \left(\phi_{yz} - \phi_{zy}\right)\hat{\mathbf{i}} + (\phi_{zx} - \phi_{xz})\hat{\mathbf{j}} + \left(\phi_{xy} - \phi_{yx}\right)\hat{\mathbf{k}}. \quad (14)$$

Return to Eq. (13), write out the double sum, and use wedge product antisymmetry to find

$$\partial\wedge(\partial\wedge\phi) = \left(\phi_{yz} - \phi_{zy}\right)(d\vec{y}\wedge d\vec{z}) + (\phi_{zx} - \phi_{xz})(d\vec{z}\wedge d\vec{x})$$

$$+\left(\phi_{xy} - \phi_{yx}\right)(d\vec{x}\wedge d\vec{y}). \quad (15)$$

Because the geometric interpretation of $(d\vec{x}\wedge d\vec{y})$ corresponds to $\hat{\mathbf{k}}$ and similarly for the other directed bivectors, we see that Eqs. (13) and (15) are equivalent and vanish together.

Turning to $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, the exterior derivative of the 1-form $\tilde{\omega} = A_i d\vec{x}^i$ produces the 2-form $\partial\wedge\tilde{\omega} = \partial_j A_i (d\vec{x}^j \wedge d\vec{x}^i)$, and the second derivative a 3-form,

$$\partial \wedge (\partial\wedge\tilde{\omega}) = \frac{\partial^2 A_i}{\partial x^k \partial x^j}(d\vec{x}^k \wedge d\vec{x}^j \wedge d\vec{x}^i), \quad (16)$$

which again vanishes, as Poincaré's lemma predicts. It remains to show that Eq. (16) contains the operator $\nabla \cdot (\nabla \times \mathbf{A})$, an exercise I leave to the reader.

Applications to Electrodynamics

Consider the homogeneous Maxwell equations of electrodynamics: the Faraday-Lenz law $\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = 0$ and Gauss's law for $\mathbf{B}$, $\nabla \cdot \mathrm{B} = 0$, where $\mathbf{E}$ and $\mathbf{B}$, respectively, are the electric and magnetic fields due to all sources. Both equations are subsumed into a single one with exterior derivatives. To carry this out, introduce a timelike basis vector $d\vec{t} \equiv d\vec{x}^0$ into the 2-form,

$$\tilde{\tilde{\alpha}} \equiv E_k(d\vec{x}^k \wedge d\vec{x}^0) + \sum_{ijk\ cyclic} B_i\left(d\vec{x}^j \wedge d\vec{x}^k\right), \quad (17)$$

where the ijk indices, which run over 1, 2, 3, denote spatial dimensions. Set $\partial\wedge\tilde{\tilde{\alpha}} = 0$ to obtain

$$0 = \sum_{ijk\ cyclic} [\partial_0 B_i + (\nabla \times \mathbf{E})_i]\left(d\vec{x}^j \wedge d\vec{x}^k \wedge d\vec{x}^0\right)$$

$$+[\nabla \cdot \mathbf{B}](d\vec{x}^1 \wedge d\vec{x}^2 \wedge d\vec{x}^2). \quad (18)$$

Because the two trivectors in Eq. (18) are independent of one another, the coefficients of each one must be set to zero, producing the Faraday-Lenz and magnetic Gauss's laws.

For the inhomogeneous Maxwell equations, define a pair of 2-forms, one with the magnetic field $\mathbf{H}$ and the electric field $\mathbf{D}$ respectively produced by free electric currents and free charges:

$$\tilde{\tilde{\beta}} \equiv -H_i(d\vec{x}^i \wedge d\vec{t}) + \sum_{ijk\ cyclic} D_i \vec{x}^j \wedge d\vec{x}^k). \quad (19)$$

Let the other 2-form include the current density $\mathbf{J}$ and charge density ρ for free electric charges:

$$\tilde{\tilde{\gamma}} \equiv \sum_{ijk\ cyclic} J_i\,(\vec{x}^j \wedge d\vec{x}^k \wedge d\vec{t}) - \rho(d\vec{x}\wedge d\vec{y}\wedge d\vec{z}). \quad (20)$$

By setting $\partial\wedge\tilde{\tilde{\beta}} = -\tilde{\tilde{\gamma}}$ and equating coefficients of like 3-forms, the inhomogeneous Maxwell equations emerge: the Amperé-Maxwell law,

$$\nabla \times \mathbf{H} + \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}, \quad (21)$$

and Gauss's Law for $\mathbf{D}$,

$$\nabla \cdot \mathbf{D} = \rho. \quad (22)$$

Integrals of r-Forms

Because the derivative of a differential r-form produces a $(r+1)$-form, we expect the integral of a differential r-form to yield a $(r-1)$-form for $r \geq 1$. Here it is useful to replace the notation $\partial\wedge T$ with

$\mathbf{d}T$. One might expect the integral of $\mathbf{d}T$ to be merely T, but in the literature on differential forms we find another weird-looking expression:

$$\int_{\mathcal{R}} \mathbf{d}T = \int_{bd\,\mathcal{R}} T, \tag{23}$$

where $bd\,\mathcal{R}$ denotes the boundary of the higher-dimensioned region $\mathcal{R}$. The appearance of T still within an integral on the right-hand side of Eq. (23) can be unsettling, but recall that a differential r-form for $r > 0$ already contains one or more differentials, such as $\tilde{T} = A_\mu d\vec{x}^\mu$. A few comments might enhance familiarity. Recall that in ordinary calculus, we write

$$\int_a^b df(x) = f(b) - f(a), \tag{24}$$

where the antiderivative of df is evaluated at the endpoints of the interval $[a,b]$—the boundary of the integration interval. A higher-dimension analog arises in Gauss's divergence theorem,

$$\iiint_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{A})\, dxdydz = \oint_{\mathcal{S}} \mathbf{A} \cdot \hat{\mathbf{n}}\; d\mathcal{S}, \tag{25}$$

where $\mathcal{S}$ denotes the closed surface that forms the boundary of volume $\mathcal{V}$. Another example arises in Stoke's theorem,

$$\iint_{\mathcal{S}'} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \hat{\mathbf{n}}\; dS' = \oint_{\mathcal{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{r}, \tag{26}$$

where $\mathcal{C}$ denotes the closed contour forming the boundary of the surface $\mathcal{S}'$. Equations (24)-(26) are merely special cases of Eq. (23)! The terse notation in Eq. (23) comes to life with an example. Consider, say, a differential 1-form $\tilde{\psi} = A_\mu d\vec{x}^\mu$. Evaluate its exterior derivative:

$$\begin{aligned} \mathbf{d}\tilde{\psi} &= \partial_\nu A_\mu\, (d\vec{x}^\nu \wedge d\vec{x}^\mu) \\ &= \sum_{\mu<\nu} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) (d\vec{x}^\mu \wedge d\vec{x}^\nu). \end{aligned} \tag{27}$$

Undo the derivative by integrating $\mathbf{d}\psi$ over a region $\mathcal{R}$. According to Eq. (23) we are to write

$$\int_{\mathcal{R}} \mathbf{d}\tilde{\psi} = \int_{bd\,\mathcal{R}} \tilde{\psi}. \tag{28}$$

To bring this to life, in three-dimensional Euclidean space, let $\mathcal{R}$ be an unclosed surface $\mathcal{S}'$ that has closed contour $\mathcal{C}$ for its boundary. Use Eq. (27) on the left-hand side of Eq. (28) and use $\tilde{\psi} = A_\mu d\vec{x}^\mu$ on the right-hand side, with μ and ν ranging through 1, 2, 3. Equation (28) then becomes

$$\int_{\mathcal{S}'} \sum_{\mu<\nu} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) (d\vec{x}^\mu \wedge d\vec{x}^\nu) = \oint_{\mathcal{C}} A_\mu d\vec{x}^\mu, \tag{29}$$

and we behold Stoke's theorem in the language of differential forms! As Steven Weinberg rightly observed, these differential forms and exterior derivatives "possess certain remarkably simple and useful properties."[2]

I hope this little introduction to differential forms and exterior derivatives will lower barriers to making their acquaintance and, as Misner, Thorne, and Wheeler intended, enable us to become "better acquainted with their charms."[3]

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References

1. References cited in Part I that are important to Part II include Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (W.H. Freeman & Co., 1973); Harley Flanders, *Differential Forms with Applications to the Physical Sciences* (Academic Press, 1963); Dwight E. Neuenschwander, *Tensor Calculus for Physics* (Johns Hopkins University Press, 2015), Ch. 8.

2. Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, 1972), pp. 113–114.

3. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (W.H. Freeman & Co., 1973), p. 45.

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