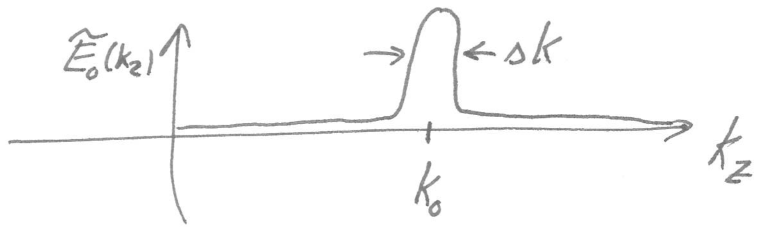
Returning to the dispersive medium, we assume that $\tilde{E}_0(k_z)$ is peaked about k_0 with a width $\Delta k \ll k_0$,



and insert the Taylor expansion

$$\omega(k_z) = \omega_0 + \omega_0'(k_z - k_0) + \frac{1}{2}\omega_0''(k_z - k_0)^2 + \cdots$$

into
$$E(z,t) = \int_{-\infty}^{\infty} \frac{dk_z}{\sqrt{2\pi}} \tilde{E}_0(k_z)\, e^{i[k_z z - \omega(k_z) t]}$$

so
$$E(z,t) \simeq e^{i(k_0\omega_0' - \omega_0)t} \int_{-\infty}^{\infty} \frac{dk_z}{\sqrt{2\pi}} \tilde{E}_0(k_z)\, e^{ik_z(z - \omega_0' t) - \frac{i}{2}\omega_0'' t (k_z - k_0)^2}$$

If we neglect ω_0'', then we again obtain a dispersionless wave form

$$E(z,t) = e^{i(k_0\omega_0' - \omega_0 t)} E_+(z - \omega_0' t)$$

where $E_+ \equiv \int_{-\infty}^{\infty} \frac{dk_z}{\sqrt{2\pi}} \tilde{E}_0(k_z)\, e^{ik_z(z - \omega_0' t)}$

and the wave form or "pulse" moves with the group velocity $\omega_0' = \left.\frac{d\omega}{dk_z}\right|_{k_z = k_0}$. The term ω_0'' causes dispersion; that is, the wave packet will spread out.

We take as a special case of $\tilde{E}_0(k_z)$ a gaussian

$$\tilde{E}_0(k_z) = \tilde{E}_0 \, e^{-\frac{(k_z-k_0)^2}{2(\Delta k)^2}} \qquad \text{then}$$

$$E(z,t) = \tilde{E}_0 \, e^{i(k_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dk_z}{\sqrt{2\pi}} \, e^{i(k_z-k_0)z} \, e^{-\frac{(k_z-k_0)^2}{2(\Delta k)^2}} \, e^{-i\omega_0' t(k_z-k_0)} \, e^{-\frac{i\omega_0''}{2}t(k_z-k_0)^2}$$

change variable $k \equiv k_z - k_0 \qquad dk = dk_z$

$$E(z,t) = \tilde{E}_0 \, e^{i(k_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \, e^{ikz} \, e^{\frac{-k^2}{2(\Delta k)^2}} \, e^{-i\omega_0' t k} \, e^{-\frac{i\omega_0''}{2} t k^2}$$

Homework: Show that this last integral is

$$e^{-\frac{(z-\omega_0' t)^2(\Delta k)^2}{2[1+t^2(\Delta k)^4(\omega_0'')^2]}} \sqrt{\frac{(\Delta k)^2}{1+i(\Delta k)^2\omega_0'' t}} \; e^{\frac{i\omega_0'' t(\Delta k)^4(z-\omega_0' t)^2}{2(1+t^2(\omega_0'')^2(\Delta k)^4)}}$$

Hint: Complete the square in k and make a contour rotation in the complex k-plane.

The essence of the result is that

$$|E(z,t)|^2 = \tilde{E}_0^{\,2} \, \frac{(\Delta k)^2}{1+(\Delta k)^4(\omega_0'')^2 t^2} \, e^{\frac{-(z-\omega_0' t)^2(\Delta k)^2}{1+(\Delta k)^4(\omega_0'')^2 t^2}}$$

disperses — i.e. spreads as a function of time. The exponential can be written as

$$e^{-\frac{(z-\omega_0' t)^2}{(\Delta z)^2}} \qquad \text{for width } \Delta z = \frac{\sqrt{1+(\Delta k)^4(\omega_0'')^2 t^2}}{\Delta k}$$

Note that $\Delta z \, \Delta k = 1 + (\Delta k)^4(\omega_0'')^2 t^2 \geq 1$.

The relation $\omega(k_z)$ is called the dispersion relation for a wave. Examples are all throughout physics:

de Broglie wave: $\omega = \frac{\hbar}{2m} k^2$

plasma wave: $\omega = \sqrt{\omega_p^2 + k^2 c^2}$ (see next section)

Unless $\omega = kv$ (for constant v), a wave packet for the type of wave considered <u>must disperse</u>. However, if you have a non-linear wave, the non-linearity can compensate for the dispersion and non-dispersive wave packets can propagate in the dispersive medium. These waves are loosely referred to as <u>solitons</u>.

E.g. The non-linear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - D|\psi|^2 \psi$$

has a soliton solution

$$\psi(x,t) = A\, e^{-i\omega t}\, e^{\frac{imvx}{\hbar}} \operatorname{sech}\left(\frac{x - vt}{\Delta}\right)$$

where v is the wave-packet velocity, $\hbar\omega$ is the soliton energy, D is the potential, and Δ is the soliton width.

You can solve for A, $\hbar\omega$, and Δ.

Free Electron Gas

We now consider the case of a free electron gas — that is, there are no bound electrons. Since we want the system to be electrically neutral, we take a uniform background of positive charge which compensates for the electrons. This positive background represents massive ions which do not respond to the electromagnetic wave because of their inertia. We set $\frac{\epsilon}{\epsilon_0} = 1 = \frac{\mu}{\mu_0}$ but now we have a charge density $\rho(\vec{r},t)$ and a current density $\vec{J}(\vec{r},t)$. For a monochromatic plane wave, we have:

$$\rho(\vec{r},t) = Re\left[\rho_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\vec{J}(\vec{r},t) = Re\left[\vec{J}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$i\vec{k}\cdot\vec{E}_0 = \frac{\rho_0}{\epsilon_0}$$

$$\vec{k}\cdot\vec{B}_0 = 0$$

$$\vec{k}\times\vec{E}_0 = \omega\vec{B}_0$$

$$i\vec{k}\times\vec{B}_0 = \mu_0\vec{J}_0 - \frac{i\omega}{c^2}\vec{E}_0$$

We construct a model for the electron gas in which electrons are driven by the electric field of the wave but are damped in their motion by collisions with the neutralizing positive ions.

For the a^{th} electron, the equation of motion is

$$m\ddot{\vec{r}}_a + m\gamma\dot{\vec{r}}_a = -e\vec{\mathcal{E}} = -e\,Re\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}_a - \omega t)}\right]$$

This again is a non-linear equation so we assume that the amplitude of the electron's motion is small compared to the wavelength and replace $e^{i\vec{k}\cdot\vec{r}_a}$ by $e^{i\vec{k}\cdot\langle\vec{r}_a\rangle}$ where $\langle\vec{r}_a\rangle$ is the average location of the a^{th} electron. Introducing a complex position vector $\vec{z}_a$ with $\vec{r}_a = Re[\vec{z}_a]$ we have

$$m\ddot{\vec{z}}_a + m\gamma\dot{\vec{z}}_a = -eE_0\, e^{i(\vec{k}\cdot\langle\vec{r}_a\rangle - \omega t)}$$

We find

$$\vec{z}_a = -\frac{e}{m}\vec{E}_0 \frac{e^{i(\vec{k}\cdot\langle\vec{r}_a\rangle - \omega t)}}{-\omega^2 - i\gamma\omega}$$

and the velocity is

$$\dot{\vec{z}}_a = \frac{-i\frac{e}{m}\vec{E}_0}{\omega + i\gamma}\, e^{i(\vec{k}\cdot\langle\vec{r}_a\rangle - \omega t)}$$

We consider a small volume ΔV with dimension $L \ll \lambda$. Then the current density is

$$\vec{J} = \frac{\sum\limits_{\substack{\text{electrons}\\ \text{in } \Delta V}} (-e)\dot{\vec{z}}_a}{\Delta V} \longrightarrow \frac{Ne^2}{m}\frac{1}{\gamma - i\omega}\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\vec{r}$ is the center of the small volume ΔV.

Thus we have $\vec{J} = \sigma \vec{E}$ where the conductivity σ is given by
$$\sigma = \frac{Ne^2}{m} \frac{1}{\gamma - i\omega}$$

For $\omega \ll \gamma$, the conductivity is real and leads to Joule heating.

The Ampère law becomes

$$i\vec{k} \times \vec{B}_0 = \mu_0 \frac{Ne^2}{m} \frac{1}{\gamma - i\omega} \vec{E}_0 - \frac{i\omega}{c^2} \vec{E}_0$$

$$= \left[\frac{\omega_p^2}{c^2} \frac{1}{\gamma - i\omega} - \frac{i\omega}{c^2} \right] \vec{E}_0$$

where the plasma frequency is $\omega_p \equiv \sqrt{\frac{Ne^2}{\epsilon_0 m}}$

Since $\vec{k} \cdot (\vec{k} \times \vec{B}_0) = 0$, we have

$$\vec{k} \cdot \vec{E}_0 \left[\frac{\omega_p^2}{c^2} \frac{1}{\gamma - i\omega} - \frac{i\omega}{c^2} \right] = 0$$

Hence, either ① $\frac{\omega_p^2}{c^2} \frac{1}{\gamma - i\omega} - \frac{i\omega}{c^2} = 0$ or ② $\vec{k} \cdot \vec{E}_0 = 0$

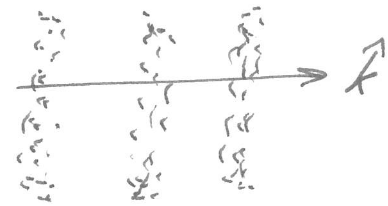
For case ①, we have $\vec{k} \times \vec{B}_0 = 0$, and since $\vec{k} \cdot \vec{B}_0 = 0$ we conclude that $\vec{B}_0 = 0$. Then by Faraday's Law, we have $\vec{k} \times \vec{E}_0 = 0$ so the electric field is longitudinal (along $\hat{k}$).

The charge density is

$$\rho(\vec{r},t) = \epsilon_0 \, Re\left[i\vec{k}\cdot\vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

for real $\vec{E}_0$ with $\rho_0 \equiv \epsilon_0 \vec{k}\cdot\vec{E}_0$

$$\rho(\vec{r},t) = \rho_0 \sin(\omega t - \vec{k}\cdot\vec{r})$$



For negligible damping we also find ($\gamma \ll \omega$)

$$\omega_p^2 - \omega^2 = 0 \quad \text{or} \quad \boxed{\omega = \omega_p}$$

Note that such a wave has no group velocity $\frac{d\omega}{dk} = 0$ and hence cannot describe a moving wave packet.

For case (2) we have $\vec{k}\cdot\vec{E}_0 = 0$. Substitute Faraday's Law into Ampère's law and obtain:

$$i\vec{k}\times\left(\frac{\vec{k}\times\vec{E}_0}{\omega}\right) = \frac{i}{\omega}\left[\vec{k}\,\underbrace{(\vec{k}\cdot\vec{E}_0)}_{0} - k^2\vec{E}_0\right] = \left[\frac{\omega_p^2}{c^2}\frac{1}{\gamma - i\omega} - \frac{i\omega}{c^2}\right]\vec{E}_0$$

$$\Rightarrow \frac{-ik^2}{\omega} = \frac{\omega_p^2}{c^2}\frac{1}{\gamma - i\omega} - \frac{i\omega}{c^2}$$

If $\gamma \ll \omega$, we can neglect γ and

$$k^2c^2 = \omega^2 - \omega_p^2 \quad \text{or} \quad \boxed{\omega = \sqrt{\omega_p^2 + k^2c^2}}$$

dispersion relation for free electron gas (plasma)