D) Radiation from a Moving Charge

We now determine the electromagnetic field due to a charge in an arbitrary state of motion. We assume that the charged particle is acted upon by some (generally unspecified) force which produces a trajectory $\vec{r}'(t')$. The particle has a 4-current density $J^\mu(x)$:

$$J^0 = c\rho = qc\,\delta^{(3)}[\vec{r} - \vec{r}'(t')]$$

$$\vec{J} = q\,\vec{v}(t)\,\delta^{(3)}[\vec{r} - \vec{r}'(t')]$$

where $\vec{v}'(t') = \dfrac{d\vec{r}'(t')}{dt'} = \dot{\vec{r}}'(t')$. We will find the fields by means of the potentials Φ and $\vec{A}$. We use them in Lorentz gauge for which

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \qquad \text{and}$$

$$\left.\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho \\ \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} &= \frac{4\pi}{c}\vec{J} \end{aligned}\right\} \quad \text{c.g.s units}$$

In 4-vector notation, we have $A^\mu = (\Phi, \vec{A})$

$$\partial_\mu A^\mu = 0 \qquad \text{and} \qquad \Box^2 A^\mu = \frac{4\pi}{c} J^\mu$$

where $\Box^2 = \partial_\nu \partial^\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is the d'Alembertian scalar differential operator. The Lorentz gauge is manifestly covariant - the same gauge condition holds in all inertial frames. This is not true of the Coulomb (or radiation) gauge. Although it is possible to use Coulomb gauge, it is far better to use Lorentz gauge to make systematic approximations (for example, in quantum electrodynamics, QED).

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We solve the wave equation $\Box^2 A^\mu = \frac{4\pi}{c} J^\mu$ by means of a Green function with suitable boundary condition – that of <u>outgoing</u> waves. This is the <u>retarded</u> Green function which obeys:

- $G_{ret}(x-x') = 0$ if $t < t'$
- $\Box^2 G_{ret}(x-x') = 4\pi c\, \delta^4(x-x')$ ← x^μ space + time

with solution

$$G_{ret}(x-x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{G}_\omega(\vec{r}-\vec{r}')\, e^{-\frac{i\omega}{c}(x^0 - x^{0\prime})}$$

(Fourier transform time part only)

where

$$\tilde{G}_\omega(\vec{r}-\vec{r}') = \frac{e^{\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

Thus

$$G_{ret}(x-x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-\frac{i\omega}{c}\left[x^0 - x^{0\prime} - |\vec{r}-\vec{r}'|\right]}}{|\vec{r}-\vec{r}'|}$$

$$= \frac{c\,\delta(x^0 - x^{0\prime} - |\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} = \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

(Remember $\delta(ax) = \frac{1}{a}\delta(x)$)

We can also write the Green function in an explicitly Lorentz invariant form by noting that $x^0 > x^{0\prime}$ under a general (isochronous) Lorentz transformation. The time-ordering of two events on the light-cone can't be changed.

$$G_{ret}(x-x') = c\,\theta(x^0 - x^{0\prime})\,\frac{\delta(x^0 - x^{0\prime} - |\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}$$

$$= 2c\,\theta(x^0 - x^{0\prime})\,\delta\Big(2\,|\vec{r}-\vec{r}\,'|\,(x^0 - x^{0\prime} - |\vec{r}-\vec{r}\,'|)\Big)$$

$$= 2c\,\theta(x^0 - x^{0\prime})\,\delta\Big[(x^0 - x^{0\prime} + |\vec{r}-\vec{r}\,'|)(x^0 - x^{0\prime} - |\vec{r}-\vec{r}\,'|)\Big]$$

(since $x^0 - x^{0\prime} = |\vec{r}-\vec{r}\,'|$)

$$= 2c\,\theta(x^0 - x^{0\prime})\,\delta\Big[(x^0 - x^{0\prime})^2 - (\vec{r}-\vec{r}\,')^2\Big]$$

$$= 2c\,\theta(x^0 - x^{0\prime})\,\delta\Big[(x^\mu - x^{\mu\prime})(x_\mu - x_\mu')\Big]$$

which is manifestly Lorentz invariant.

Recall that the significance of $G_{ret}(x-x')$ is that a source is initiated at location $\vec{r}\,'$ at time t' and a response is felt at location $\vec{r}$ at time

$$t = t' + \underbrace{\frac{|\vec{r}-\vec{r}\,'|}{c}}_{\text{pulse travel time}}$$

The 4-vector potential is then given by

$$A^\mu(x) = A^\mu_h(x) + \int d^4x'\, G_{ret}(x-x')\, \frac{J^\mu(x')}{c^2}$$

where $A^\mu_h(x)$ is a solution to the homogeneous equation $\Box^2 A^\mu_h(x) = 0$

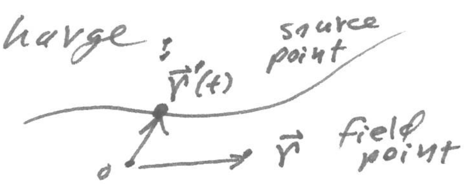
Proof:

$$\Box^2 A^\mu(x) = 0 + \int d^4x'\, \Box^2 G_{ret}(x-x')\, \frac{J^\mu(x')}{c^2}$$

$$= \int d^4x'\, 4\pi c\, \delta^4(x-x')\, \frac{J^\mu(x')}{c^2} = \frac{4\pi}{c} J^\mu(x) \quad \blacksquare$$

From now on, we drop the term $A^\mu_h(x)$ (we would need it for scattering).

Remember, for a moving point charge:



$$J^0(x) = q c\, \delta^3[\vec r - \vec r'(t)]$$

$$\vec J(x) = q\, \vec v'(t)\, \delta^3[\vec r - \vec r'(t)] \,, \quad \vec v'(t) = \frac{d\vec r'(t)}{dt}$$

or $$J^\mu(x) = q\, \delta^3(\vec r - \vec r'(t))\, \frac{dx'^\mu(t)}{dt}$$

$$A^\mu(x) = A^\mu_h(x) + \int d^4x''\, G_{ret}(x-x'')\, \frac{J^\mu(x'')}{c^2}$$

need dummy integration variable x''

$$= 0 + \int_{-\infty}^{\infty} dt'' \int d^3x''\, \frac{\delta\left(t - t'' - \frac{|\vec{r}-\vec{r}''|}{c}\right)}{|\vec{r}-\vec{r}''|}\, \frac{q\,\delta^3[\vec{r}'' - \vec{r}'(t'')]\,\frac{dx'^\mu(t'')}{dt''}}{c}$$

Use the $\delta^3[\vec{r}'' - \vec{r}'(t'')]$ delta function to perform the $\int d^3x''$ integration. $\vec{r}'' \to \vec{r}'(t'')$

Define $f(t'') = t'' - t + \dfrac{|\vec{r} - \vec{r}'(t'')|}{c}$

argument of the one remaining delta function

Now, the only dummy integration variable is t''. Call it t' instead: $t'' \to t'$

$$f(t') = t' - t + \frac{|\vec{r} - \vec{r}'(t')|}{c}$$

$$A^\mu(x) = \frac{q}{c} \int_{-\infty}^{\infty} dt'\, \frac{dx'^\mu(t')}{dt'}\, \frac{\delta[f(t')]}{|\vec{r} - \vec{r}'(t')|}$$

Used $\delta(-f) = \delta(f)$

$$= \frac{q}{c} \int df\, \frac{1}{\left(\frac{df(t')}{dt'}\right)}\, \frac{dx'^\mu(t')}{dt'}\, \frac{\delta[f(t')]}{|\vec{r} - \vec{r}'(t')|}$$

eq. (1)

$$A^\mu(x) = \frac{q}{c}\left[\frac{\left(\frac{dx'^\mu(t')}{dt'}\right)}{\left(\frac{df(t')}{dt'}\right)|\vec{r}-\vec{r}'(t')|}\right]_{f=0}$$

$f=0$ means $t = t' + \frac{|\vec{r}-\vec{r}'(t')|}{c}$. Thus when the particle is at location $\vec{r}'(t')$ at time t'; the field will be felt at location $\vec{r}$ and time t.

We call $t' = t - \frac{|\vec{r}-\vec{r}'(t')|}{c}$ the retarded time. This relation is highly implicit. Also note that "observer time" t and "source time" t' (the retarded time) are measured in the same frame of reference.

$$\frac{df(t')}{dt'} = \frac{d}{dt'}\left[t' - t + \frac{|\vec{r}-\vec{r}'(t')|}{c}\right]$$

$$= 1 - 0 + \frac{1}{c}\frac{\partial}{\partial t'}|\vec{r}-\vec{r}'(t')|$$

Look at the last term

$$\frac{\partial}{\partial t'}|\vec{r}-\vec{r}'(t')| = \frac{\partial}{\partial t'}\sqrt{[\vec{r}-\vec{r}'(t')]\cdot[\vec{r}-\vec{r}'(t')]}$$

$$= \frac{1}{2\sqrt{[\vec{r}-\vec{r}'(t')][\vec{r}-\vec{r}'(t')]}}\frac{\partial}{\partial t'}[\vec{r}-\vec{r}'(t')]^2$$

$$= \frac{1}{2|\vec{r}-\vec{r}'(t')|}\frac{\partial}{\partial t'}\left[r^2 - 2\vec{r}\cdot\vec{r}'(t') + \vec{r}'(t')\cdot\vec{r}'(t')\right]$$

$$= -\frac{\vec{r}-\vec{r}'(t')}{|\vec{r}-\vec{r}'(t')|}\cdot\vec{v}'(t') = -\hat{n}(t')\cdot\vec{v}'(t')$$

where $\hat{n}(t') \equiv \frac{\vec{r}-\vec{r}'(t')}{|\vec{r}-\vec{r}'(t')|}$ and $\vec{v}'(t') \equiv \frac{d\vec{r}'(t')}{dt'}$

Thus $\frac{df}{dt'} = 1 - \frac{\hat{n}\cdot\vec{v}'}{c}$ everything has t' functional dependence

$$A^\mu(x) = \frac{\frac{q}{c}\frac{dx'^\mu(t')}{dt'}}{\left(1-\frac{\hat{n}\cdot\vec{v}}{c}\right)|\vec{r}-\vec{r}'(t')|} \qquad t' = t - \frac{|\vec{r}-\vec{r}'(t')|}{c}$$

Liénard - Wiechert Potentials

$$\left.\begin{aligned}
\Phi(\vec{r},t) &= \frac{q}{\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)\left|\vec{r}-\vec{r}'(t')\right|} \\[2em]
\vec{A}(\vec{r},t) &= \frac{q\,\frac{\vec{v}'(t')}{c}}{\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)\left|\vec{r}-\vec{r}'(t')\right|}
\end{aligned}\right\}$$

Evaluated at $t' = t - \frac{|\vec{r}-\vec{r}'(t')|}{c}$ retarded time

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We write these as

$$\Phi(\vec{r},t) = \frac{q}{R(\vec{r},t')} \qquad \vec{A}(\vec{r},t) = \frac{q\,\frac{\vec{v}'(t')}{c}}{R(\vec{r},t')}$$

where $R \equiv |\vec{r}-\vec{r}'(t')| - [\vec{r}-\vec{r}'(t')]\cdot\frac{\vec{v}'(t')}{c}$

To find the $\vec{E}$ and $\vec{B}$ fields, we need the following relations:

$$\vec{\nabla} t' = \vec{\nabla}\left[t - \frac{|\vec{r}-\vec{r}'(t')|}{c}\right] = -\vec{\nabla}\,\frac{|\vec{r}-\vec{r}'(t')|}{c}$$

total $\vec{r}$ dependence, explicit and implicit

$$\vec{\nabla} t' = -\left(\vec{\nabla}_{\vec{r}} \frac{|\vec{r}-\vec{r}\,'(t')|}{c}\right)_{\vec{r}\,'=\text{const}} - \left(\vec{\nabla}_{\vec{r}\,'} \frac{|\vec{r}-\vec{r}\,'(t')|}{c}\right)_{\vec{r}=\text{const}} \cdot \frac{\partial \vec{r}\,'(t')}{\partial t'} \vec{\nabla} t'$$

(chain rule differentiation)

But $\left(\vec{\nabla}_{\vec{r}} \frac{|\vec{r}-\vec{r}\,'(t')|}{c}\right)_{\vec{r}\,'=\text{const}} = \frac{\hat{n}}{c}$, $\frac{\partial \vec{r}\,'(t')}{\partial t'} = \vec{v}\,'(t')$,

and $\left(\vec{\nabla}_{\vec{r}\,'} \frac{|\vec{r}-\vec{r}\,'(t')|}{c}\right)_{\vec{r}=\text{const}} = -\frac{\hat{n}}{c}$

So, solving for $\vec{\nabla} t'$ we get

$$\boxed{\vec{\nabla} t' = \frac{-\frac{\hat{n}}{c}}{1 - \frac{\hat{n}\cdot\vec{v}\,'}{c}}}$$

We will also need the following: Remember $t' = t - \frac{|\vec{r}-\vec{r}\,'(t')|}{c}$

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c}\left(\vec{\nabla}_{\vec{r}\,'} |\vec{r}-\vec{r}\,'(t')|\right)_{\vec{r}=\text{const}} \cdot \vec{v}\,'(t') \frac{\partial t'}{\partial t} = 1 + \frac{\hat{n}\cdot\vec{v}\,'}{c} \frac{\partial t'}{\partial t}$$

Or, solving for $\frac{\partial t'}{\partial t}$,

$$\boxed{\frac{\partial t'}{\partial t} = \frac{1}{1 - \frac{\hat{n}\cdot\vec{v}\,'}{c}}}$$

R depends on $(\vec{r}, t')$ not $(\vec{r}, t)$ so

$$\boxed{\left(\vec{\nabla} R\right)_{t'=\text{const}} = \hat{n} - \frac{\vec{v}'}{c}}$$

$$\boxed{\frac{\partial R}{\partial t'} = -\hat{n}\cdot\vec{v}' + \frac{(v')^2}{c} - \left[\vec{r} - \vec{r}'(t')\right]\cdot\frac{\vec{a}'}{c}}$$

where $\vec{a}' \equiv \dfrac{\partial \vec{v}'(t')}{\partial t'}$

We are now in a position to calculate the electric and magnetic fields.

$$-\vec{\nabla}\Phi(\vec{r},t) = \frac{q}{R^2}\left[\left(\vec{\nabla} R(\vec{r},t')\right)_{t'=\text{const}} + \frac{\partial R(\vec{r},t')}{\partial t'}\vec{\nabla} t'(\vec{r},t)\right]$$

$$-\vec{\nabla}\Phi = \frac{q}{R^2}\left[\hat{n} - \frac{\vec{v}'}{c} + \left(\frac{\hat{n}\cdot\vec{v}' - \frac{(v')^2}{c} + \left[\vec{r}-\vec{r}'(t')\right]\cdot\frac{\vec{a}'}{c}}{1 - \frac{\hat{n}\cdot\vec{v}'}{c}}\right)\frac{\hat{n}}{c}\right]$$

$t =$ constant

$$-\frac{1}{c}\left(\frac{\partial \vec{A}(\vec{r},t)}{\partial t}\right)_{\vec{r}=const} = -\frac{1}{c}\left[\frac{\partial}{\partial t'}\,\frac{q\,\frac{\vec{v}'(t')}{c}}{R(\vec{r},t')}\right]_{\vec{r}=const}\left(\frac{\partial t'}{\partial t}\right)_{\vec{r}=const}$$

$$= \frac{q\,\frac{\vec{v}'}{c^2}}{R^2}\left(\frac{\partial R(\vec{r},t')}{\partial t'}\right)_{\vec{r}=const}\left(\frac{\partial t'(\vec{r},t)}{\partial t}\right)_{\vec{r}=const} - \frac{q\,\vec{a}'}{c^2 R}\left(\frac{\partial t'(\vec{r},t)}{\partial t}\right)_{\vec{r}=const}$$

So

$$\boxed{\begin{aligned}-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} &= \frac{q\,\vec{v}'}{c^2 R^2}\left[-\hat{n}\cdot\vec{v}' + \frac{(v')^2}{c} - [\vec{r}-\vec{r}'(t')]\cdot\frac{\vec{a}'}{c}\right]\frac{1}{1-\frac{\hat{n}\cdot\vec{v}'}{c}} \\ &\quad - \frac{q\,\vec{a}'}{c^2 R}\,\frac{1}{1-\frac{\hat{n}\cdot\vec{v}'}{c}}\end{aligned}}$$

Now assemble the pieces:

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi(\vec{r},t) - \frac{1}{c}\frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$$

The electric field is

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi(\vec{r},t) - \frac{1}{c}\frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

$$= \frac{q}{R^2}\left[\hat{n} - \frac{\vec{v}'}{c} + \left(\frac{\hat{n}\cdot\vec{v}' - \frac{(v')^2}{c} + [\vec{r}-\vec{r}'(t')]\cdot\frac{\vec{a}'}{c}}{1-\frac{\hat{n}\cdot\vec{v}'}{c}}\right)\frac{\hat{n}}{c}\right]$$

$$+\frac{q\vec{v}'}{c^2R^2}\left[-\hat{n}\cdot\vec{v}' + \frac{(v')^2}{c} - [\vec{r}-\vec{r}'(t')]\cdot\frac{\vec{a}'}{c}\right]\frac{1}{1-\frac{\hat{n}\cdot\vec{v}'}{c}} - \frac{q\vec{a}'}{c^2R}\frac{1}{1-\frac{\hat{n}\cdot\vec{v}'}{c}}$$

Remember: $R \equiv |\vec{r}-\vec{r}'(t')| - [\vec{r}-\vec{r}'(t')]\cdot\frac{\vec{v}'(t')}{c}$

$$R = |\vec{r}-\vec{r}'(t')|\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)$$

$$\vec{E}(\vec{r},t) = \frac{q}{|\vec{r}-\vec{r}'(t')|^2}\frac{1}{\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)^3}\left[\left(\hat{n}-\frac{\vec{v}'}{c}\right)\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right) + \left(\hat{n}\cdot\vec{v}' - \frac{(v')^2}{c}\right)\frac{\hat{n}}{c}\right.$$

$\uparrow$ from $\nabla\Phi$

$$\left. - \frac{\vec{v}'}{c^2}(\hat{n}\cdot\vec{v}') + \frac{\vec{v}'}{c^2}\frac{(v')^2}{c}\right]$$

$\leftarrow$ from $\frac{\partial A}{\partial t}$

$$+\frac{q}{|\vec{r}-\vec{r}'(t')|}\frac{1}{\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)^3}\left[\left(\hat{n}\cdot\frac{\vec{a}'}{c}\right)\frac{\hat{n}}{c} - \frac{\vec{a}'}{c^2} - \frac{\vec{v}'}{c^2}\left(\hat{n}\cdot\frac{\vec{a}'}{c}\right) + \frac{\vec{a}'}{c^2}\left(\hat{n}\cdot\frac{\vec{v}'}{c}\right)\right]$$

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the first set of square brackets above [...] is

$$\left[\hat{n} - \vec{\beta} - \underline{\hat{n}(\hat{n}\cdot\vec{\beta})} + \underline{\vec{\beta}(\hat{n}\cdot\vec{\beta})} + \underline{\hat{n}(\hat{n}\cdot\vec{\beta})} - \hat{n}\beta^2 - \underline{\vec{\beta}(\hat{n}\cdot\vec{\beta})} + \vec{\beta}\beta^2\right]$$

underlined terms cancel $= [(\hat{n}-\vec{\beta})(1-\beta^2)]$

$$\vec{E}(\vec{r},t) = \vec{E}_{con}(\vec{r},t) + \vec{E}_{rad}(\vec{r},t)$$

$$\vec{E}_{con}(\vec{r},t) = \frac{q\left(\hat{n} - \frac{\vec{v}'}{c}\right)}{\left|\vec{r} - \vec{r}'(t')\right|^2} \frac{1 - \frac{(v')^2}{c^2}}{\left(1 - \frac{\hat{n}\cdot\vec{v}'}{c}\right)^3}$$

"Convection Field"
also called velocity field

$$\vec{E}_{rad}(\vec{r},t) = \frac{q}{c} \frac{\hat{n}\times\left[\left(\hat{n} - \frac{\vec{v}'}{c}\right)\times\frac{\vec{a}'}{c}\right]}{\left|\vec{r} - \vec{r}'(t')\right| \left(1 - \frac{\hat{n}\cdot\vec{v}'}{c}\right)^3}$$

"Radiation Field"
also called acceleration field

Remember $\vec{A}(\vec{r},t) = \dfrac{q\frac{\vec{v}'(t')}{c}}{R(\vec{r},t')}$

but t' depends on $\vec{r}$ implicitly

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$$

$$= q\left(\vec{\nabla}\frac{1}{R}\right)_{t'}\times\frac{\vec{v}'}{c} + q\left(\frac{\partial}{\partial t'}\frac{1}{R}\right)_{\vec{r}}\left(\vec{\nabla}t'\right)_t\times\frac{\vec{v}'}{c} + \frac{q}{R}\left(\vec{\nabla}t'\right)_t\times\frac{\vec{a}'}{c}$$

$$= -\frac{q}{R^2}\left(\hat{n} - \frac{\vec{v}'}{c}\right)\times\frac{\vec{v}'}{c} + \frac{q}{R^2}\left[-\hat{n}\cdot\vec{v}' + \frac{(v')^2}{c} - \left[\vec{r} - \vec{r}'(t')\right]\cdot\frac{\vec{a}'}{c}\right]\frac{\frac{\hat{n}}{c}\times\frac{\vec{v}'}{c}}{1 - \frac{\hat{n}\cdot\vec{v}'}{c}}$$

$$- \frac{q}{R}\frac{\frac{\hat{n}}{c}\times\frac{\vec{a}'}{c}}{1 - \frac{\hat{n}\cdot\vec{v}'}{c}}$$

$$= \frac{q}{\left|\vec{r} - \vec{r}'(t')\right|^2}\frac{1}{\left(1 - \frac{\hat{n}\cdot\vec{v}'}{c}\right)^3}\left[-\frac{\hat{n}\times\vec{v}'}{c}\left(1 - \frac{(v')^2}{c^2}\right)\right]$$

$$+ \frac{q}{\left|\vec{r} - \vec{r}'(t')\right|}\frac{1}{\left(1 - \frac{\hat{n}\cdot\vec{v}'}{c}\right)^3}\frac{1}{c}\left[-\hat{n}\times\frac{\vec{v}'}{c}\left(\hat{n}\cdot\frac{\vec{a}'}{c}\right) - \hat{n}\times\frac{\vec{a}'}{c}\left(1 - \frac{\hat{n}\cdot\vec{v}'}{c}\right)\right]$$

$$\vec{B}(\vec{r},t) = \hat{n} \times \vec{E}(\vec{r},t) = \hat{n} \times \vec{E}_{con}(\vec{r},t) + \hat{n} \times \vec{E}_{rad}(\vec{r},t)$$

$$= \vec{B}_{con}(\vec{r},t) + \vec{B}_{rad}(\vec{r},t)$$

While it is also true that $\vec{E}_{rad} = \vec{B}_{rad} \times \hat{n}$,

notice that $\vec{E}_{con} \neq \vec{B}_{con} \times \hat{n}$.

The radiation fields fall off as $\dfrac{1}{|\vec{r} - \vec{r}'(t')|}$

while the convection fields fall off as $\dfrac{1}{|\vec{r} - \vec{r}'(t')|^2}$

Remember that $\vec{r}'(t')$, $\hat{n}(t')$, $\vec{v}'(t')$, and $\vec{a}'(t')$

all depend on the retarded time $t' = t - \dfrac{|\vec{r} - \vec{r}'(t')|}{c}$.

For $v' \ll c$ we have

$$\vec{E}_{con}(\vec{r},t) = \frac{q\,\hat{n}}{|\vec{r} - \vec{r}'(t')|^2} \qquad \vec{E}_{rad}(\vec{r},t) = \frac{q\,\hat{n} \times \left(\hat{n} \times \frac{\vec{a}'}{c^2}\right)}{|\vec{r} - \vec{r}'(t')|}$$

The power radiated by the particle in this non-relativistic limit is obtained by integrating the Poynting vector over a large sphere

$$P = \frac{c}{4\pi} \oint dS\, \hat{n} \cdot \left(\vec{E}_{rad} \times \vec{B}_{rad} \right) = \frac{c q^2}{4\pi} \int d\Omega \left| \hat{n} \times \left(\hat{n} \times \frac{\vec{a}'}{c^2} \right) \right|^2$$

$$= \frac{q^2}{4\pi c^3} \int d\Omega \, (\hat{n} \times \vec{a}') \cdot (\hat{n} \times \vec{a}') = \frac{q^2}{4\pi c^3} \int d\Omega \, \sin^2\theta \, |\vec{a}'|^2$$

where θ is the angle between $\hat{n}$ and $\vec{a}'$

$$\int \frac{d\Omega}{4\pi} \sin^2\theta = \frac{2}{3}$$

$$\boxed{P_{(t')} = \frac{2q^2}{3c^3} \left| \vec{a}'(t') \right|^2}$$

"Larmor's power formula"
instantaneous power radiated by point source at time t'!

As an example, consider the harmonic oscillator!

$$\left.\begin{aligned} \vec{r}'(t') &= \vec{r}_0 \cos(\omega t') \\ \vec{v}'(t') &= -\omega \vec{r}_0 \sin(\omega t') \\ \vec{a}'(t') &= -\omega^2 \vec{r}_0 \cos(\omega t') \end{aligned}\right\}$$

$$P = \frac{2q^2}{3c^3} r_0^2 \omega^4 \cos^2(\omega t')$$

$$\langle P \rangle = \frac{1}{3} p_0^2 \frac{\omega^4}{c^3} \qquad (\text{c.g.s.})$$

where $\vec{p}_0 \equiv q \vec{r}_0$ is the electric dipole moment and

$$\langle \cos^2(\omega t') \rangle = \frac{1}{2}$$

In a reference frame in which $\vec{v}'(t') = 0$, the power radiated is

$$P = \frac{dW}{dt'} = \frac{dW}{dt}$$

since $dt' = \dfrac{dt}{1 - \frac{\hat{n}\cdot\vec{v}'}{c}}$

$= dt$ (if $\vec{v}' = 0$)

where W is the energy radiated by the source.

$$dW = \frac{2}{3}\frac{q^2}{c^3}|\vec{a}'|^2\, dt'$$

Since dW is an energy, we expect it to transform as the time component of a 4-vector. Furthermore, since dt' is the time component of the dx^μ 4-vector, the Larmor formula suggests that

$$\left[\frac{2}{3}\frac{q^2}{c^3}|\vec{a}'|^2\right]_{\vec{v}'=0}$$ is a Lorentz invariant (scalar).

That is, the expression above is $Z^\mu Z_\mu$ for some 4-vector Z^μ.

Remember, the 4-velocity is

$$U^\mu = (\gamma c, \gamma \vec{v}') \qquad \text{with} \quad \gamma = \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}$$

Then

$$\frac{dU^\mu}{d\tau'} = \gamma \frac{dU^\mu}{dt'} \qquad \begin{aligned} t' &= \text{source time} \\ \tau' &= \text{proper source time} \end{aligned}$$

$$= \gamma\left[c\frac{d\gamma}{dt'},\ \vec{v}'\frac{d\gamma}{dt'} + \gamma\vec{a}'\right]$$

$$\frac{d\gamma}{dt'} = \gamma^3 \frac{\vec{v}'\cdot\vec{a}'}{c^2} \qquad \frac{1}{2}\frac{d}{dt'}\vec{v}'\cdot\vec{v}' = \vec{v}'\cdot\vec{a}'$$

$$\frac{dU^\mu}{d\tau'}\frac{dU_\mu}{d\tau'} = \gamma^2\left[c^2\gamma^6\frac{(\vec{v}'\cdot\vec{a}')^2}{c^4} - \left(\gamma^3\frac{\vec{v}'\cdot\vec{a}'}{c^2}\vec{v}' + \gamma\vec{a}'\right)^2\right]$$

$$= -\gamma^6\left[(\vec{a}')^2 - \left(\frac{\vec{v}'\times\vec{a}'}{c}\right)^2\right]$$

$= -|\vec{a}'|^2$ in the frame in which $\vec{v}'=0$ $(\gamma=1)$

$$dW = -\frac{2q^2}{3c^3}\left(\frac{dU^\mu}{d\tau'}\right)\left(\frac{dU_\mu}{d\tau'}\right)dt'$$

$$dW = \frac{2q^2}{3c^3}\gamma^6\left[(\vec{a}')^2 - \left(\frac{\vec{v}'\times\vec{a}'}{c}\right)^2\right]dt' = d(cP^0)$$

$$d\vec{P} = \frac{2q^2}{3c^3}\gamma^6\left[(\vec{a}')^2 - \left(\frac{\vec{v}'\times\vec{a}'}{c}\right)^2\right]\frac{\vec{v}'}{c^2}dt' \leftarrow$$ momentum radiated by particle in dt'

If we want the <u>angular distribution</u> of the radiated power, we must use the full Poynting vector

$$P = \frac{c}{4\pi}\int d\Omega\, |\vec{r}-\vec{r}'(t')|^2\, |\vec{E}_{rad}(\vec{r},t)|^2$$

The energy radiated between observer times t_1 and t_2 (no primes) is

$$\Delta W = \frac{c}{4\pi}\int_{t_1}^{t_2} dt \int d\Omega\, |\vec{r}-\vec{r}'(t')|^2\, |\vec{E}_{rad}(\vec{r},t)|^2$$

$$= \frac{q^2}{4\pi c^3}\int_{t_1}^{t_2} dt \int d\Omega\, \frac{|\hat{n}\times[(\hat{n}-\frac{\vec{v}'}{c})\times\vec{a}']|^2}{(1-\frac{\hat{n}\cdot\vec{v}'}{c})^6}$$

$$= \frac{q^2}{4\pi c^3}\int d\Omega \int_{t_1'}^{t_2'} dt' \left(\frac{dt}{dt'}\right) \frac{|\hat{n}\times[(\hat{n}-\frac{\vec{v}'}{c})\times\vec{a}']|^2}{(1-\frac{\hat{n}\cdot\vec{v}'}{c})^6}$$

$$= \int d\Omega \int_{t_1'}^{t_2'} dt' \,\frac{dP(t')}{d\Omega}$$

integral over <u>source</u> time now

$$\frac{dt}{dt'} = \left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^3}\,\frac{|\hat{n}\times[(\hat{n}-\frac{\vec{v}'}{c})\times\vec{a}']|^2}{(1-\frac{\hat{n}\cdot\vec{v}'}{c})^5}$$

differential power radiated by source

$$\text{total power} = \frac{dW}{dt'} = \int d\Omega\, \frac{dP(t')}{d\Omega}$$

So it *must* be true that the two expressions are equal:

$$\frac{dW}{dt'} = \frac{q^2}{4\pi c^3}\int d\Omega \, \frac{\left|\hat{n}\times\left[\left(\hat{n}-\frac{\vec{v}'}{c}\right)\times\vec{a}'\right]\right|^2}{\left(1-\frac{\hat{n}\cdot\vec{v}'}{c}\right)^5} = \frac{2q^2}{3c^3}\gamma'^6\left[(\vec{a}')^2-\left(\frac{\vec{v}'\times\vec{a}'}{c}\right)^2\right]$$

but this is not at all obvious!

Now we look at two special cases.

Case 1 $\quad \vec{v}' \parallel \vec{a}'$ (Bremsstrahlung = "braking radiation")

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^3}(a')^2\frac{\sin^2\theta}{\left(1-\frac{|\vec{v}'|}{c}\cos\theta\right)^5}$$

Notice that this reduces to the Larmor result when $\vec{v}'=0$

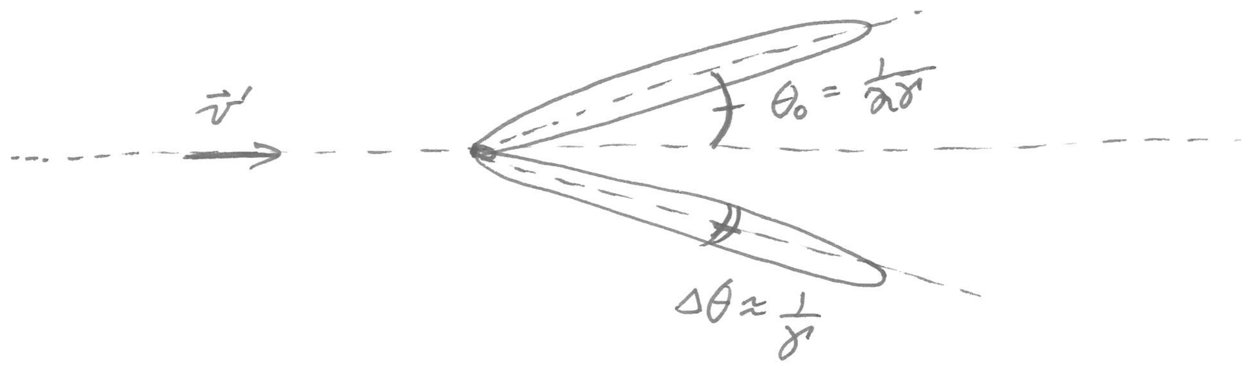
Suppose $v' \lesssim c$. Then because of the denominator factor $\left(1-\frac{\vec{v}'}{c}\cos\theta\right)^5$, the radiation is peaked forward. For small θ, we replace $\sin\theta$ by θ and $\cos\theta \approx 1-\frac{\theta^2}{2}$.

$$\frac{dP(t')}{d\Omega} \approx \frac{q^2}{4\pi c^3}(a')^2\frac{\theta^2}{\left[\left(1-\frac{v'}{c}\right)+\frac{v'}{c}\frac{\theta^2}{2}\right]^5} \approx \frac{8q^2}{\pi c^3}(a')^2\frac{\theta^2}{\left[2\left(1-\frac{v'}{c}\right)+\theta^2\right]^5}$$

where $2\left(1-\frac{v'}{c}\right) \approx \left(1+\frac{v'}{c}\right)\left(1-\frac{v'}{c}\right) = 1-\frac{(v')^2}{c^2} = \frac{1}{\gamma'^2}$

$$\frac{dP(t')}{d\Omega} \approx \frac{8q^2}{\pi c^3}\gamma'^{10}(a')^2\frac{\theta^2}{\left[1+\gamma'^2\theta^2\right]^5}$$

Now $\dfrac{\theta^2}{[1+\gamma'^2\theta^2]^5}$ peaks at $\theta_0 = \dfrac{1}{2\gamma'}$ so the differential power radiated by the particle looks like



Notice that $\vec{a}'$ can be parallel <u>or</u> antiparallel to $\vec{v}'$, acceleration <u>or</u> deceleration!

Now to find the total (not differential) power $\dfrac{dW}{dt'}$ we have three options:

1) Lorentz transformation

$$\frac{dW}{dt'} = \frac{2q^2}{3c^3}\gamma'^6\left[(a')^2 - \left(\frac{\vec{v}'\times\vec{a}'}{c}\right)^2\right] \longrightarrow \frac{2q^2}{3c^3}\gamma'^6(a')^2$$

2) integrate the exact expression $\dfrac{dP(t')}{d\Omega} = \dfrac{q^2}{4\pi c^3}(a')^2\dfrac{\sin^2\theta}{\left(1-\frac{v'}{c}\cos\theta\right)^5}$ 0 to 2π in φ, 0 to π in θ.

3) integrate the approximate expression

$$\frac{dP(t')}{d\Omega} \approx \frac{8q^2}{\pi c^3}\gamma'^{10}(a')^2\frac{\theta^2}{[1+\gamma'^2\theta^2]^5}$$ over 0 to 2π in φ, 0 to ∞ in θ

(the formula is valid only for $\theta \ll 1$, but negligible power is radiated at larger angles when $\gamma' \gg 1$.)

$$\frac{dW}{dt'} = \frac{8q^2}{\pi c^3}(a')^2\, 2\pi \int_{\theta=0}^{\infty} \theta\, d\theta \frac{\theta^2}{\left[\frac{1}{\gamma^2}+\theta^2\right]^5} = \frac{16q^2}{c^3}(a')^2 \gamma^6 \underbrace{\int_{x=0}^{\infty} dx \frac{x^3}{(1+x^2)^5}}_{\frac{1}{24}}$$

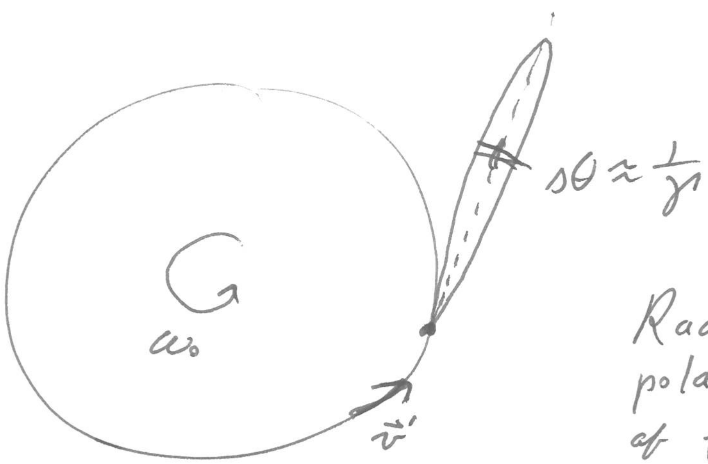
$\sin\theta\, d\theta$ for small angles

$$\frac{dW}{dt'} = \frac{2}{3}\frac{q^2}{c^3}(a')^2\gamma^6$$

<u>Case 2</u> $\quad \vec{v}' \perp \vec{a}' \quad$ (Synchrotron radiation)

$$\frac{dW}{dt'} = \frac{2q^2}{3c^3}\gamma^4 (a')^2$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{(a')^2}{\left(1-\frac{v'}{c}\cos\theta\right)^3}\left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2\left(1-\frac{v'}{c}\cos\theta\right)^2}\right]$$



Radiation is $\approx 85\%$ polarized in the plane of the orbit.

Cutoff frequency $\omega_c = 3\gamma^3\omega_0$