The fields in the incident medium are

$$\vec{E}_I(\vec{r},t) = \underbrace{\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{\text{incident}} + \underbrace{\vec{E}_0''\, e^{i(\vec{k}''\cdot\vec{r}-\omega t)}}_{\text{reflected}}$$

$$\vec{B}_I(\vec{r},t) = \frac{\vec{k}\times\vec{E}_0}{\omega}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\vec{k}''\times\vec{E}_0''}{\omega}\, e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$$

The fields in the refracted medium are

$$\vec{E}_R(\vec{r},t) = \vec{E}_0'\, e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$$

$$\vec{B}_R(\vec{r},t) = \frac{\vec{k}'\times\vec{E}_0'}{\omega}\, e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$$

When the field point $\vec{r}$ is on the boundary, we designate it by $\vec{R}$. The first important result is obtained by noting that the boundary conditions can only be satisfied if all the wave <u>phase factors</u> are the same for all $\vec{R}$ and t:

$$e^{i(\vec{k}\cdot\vec{R}-\omega t)} = e^{i(\vec{k}'\cdot\vec{R}-\omega t)} = e^{i(\vec{k}''\cdot\vec{R}-\omega t)}$$

The time aspect has already been dealt with (it resulted in the same frequency ω being used in each part of the wave). The special aspects require

$$\vec{k}\cdot\vec{R} = \vec{k}'\cdot\vec{R} = \vec{k}''\cdot\vec{R}$$

If we choose the origin on the interface, then we can write the previous equation as

$$\vec{k} \times \hat{n} = \vec{k}' \times \hat{n} = \vec{k}'' \times \hat{n} \qquad (\hat{n} \text{ is the normal})$$

or as

$$k \sin(\theta_i) = k' \sin(\theta_b) = k'' \sin(\theta_r)$$

But $k = \frac{\omega}{v} = k''$ and $k' = \frac{\omega}{v'}$

where $v = \frac{1}{\sqrt{\epsilon\mu}}$ and $v' = \frac{1}{\sqrt{\epsilon'\mu'}}$

Thus $\boxed{\theta_i = \theta_r}$ angle of incidence equals angle of reflection

and since $n = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$ = index of refraction of the incident medium

$n' = \frac{c}{v'} = \sqrt{\frac{\epsilon'\mu'}{\epsilon_0\mu_0}}$ = index of refraction of the refracted medium

$$n \sin(\theta_i) = n' \sin(\theta_b) \qquad \text{Snell's Law (or Snel's)}$$

Note that the medium with the larger index of refraction has the smaller relevant angle.

Now look again at the boundary conditions:

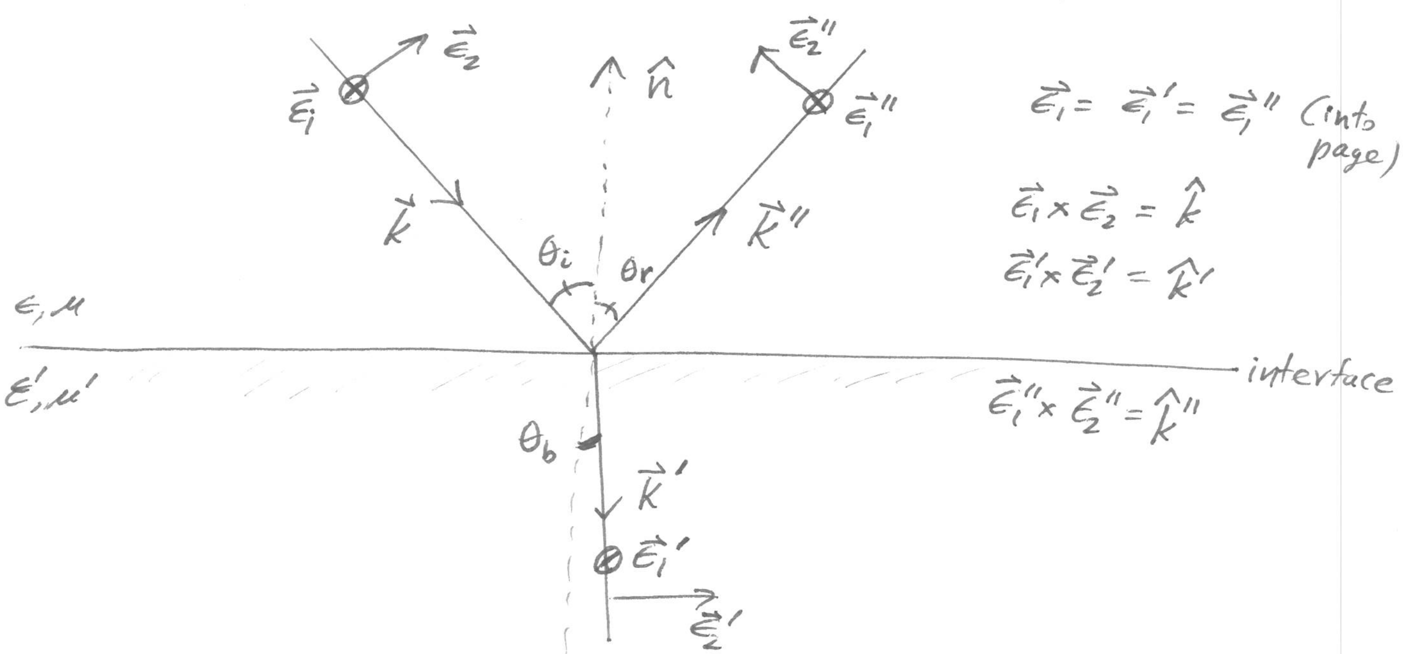
$$\hat{n}\cdot\Delta\vec{D}=0 \quad\Rightarrow\quad \hat{n}\cdot\left[\epsilon(\vec{E}_0+\vec{E}_0'')-\epsilon'\vec{E}_0'\right]=0$$

$$\hat{n}\times\Delta\vec{E}=0 \quad\Rightarrow\quad \hat{n}\times\left[\vec{E}_0+\vec{E}_0''-\vec{E}_0'\right]=0$$

$$\hat{n}\cdot\Delta\vec{B}=0 \quad\Rightarrow\quad \hat{n}\cdot\left[\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''-\vec{k}'\times\vec{E}_0'\right]=0$$

$$\hat{n}\times\Delta\vec{H}=0 \quad\Rightarrow\quad \hat{n}\times\left[\frac{\vec{k}\times\vec{E}_0}{\mu}+\frac{\vec{k}''\times\vec{E}_0''}{\mu}-\frac{\vec{k}'\times\vec{E}_0'}{\mu'}\right]=0$$

We now define the <u>plane of incidence</u> as the plane containing the vectors $\vec{k}$ and $\hat{n}$ — in our diagram, it is the plane of the paper. We also need two polarization vectors $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$:



$\vec{\epsilon}_1 = \vec{\epsilon}_1' = \vec{\epsilon}_1''$ (into page)

$\vec{\epsilon}_1\times\vec{\epsilon}_2=\hat{k}$

$\vec{\epsilon}_1'\times\vec{\epsilon}_2'=\hat{k}'$

$\vec{\epsilon}_1''\times\vec{\epsilon}_2''=\hat{k}''$

$\hat{n} \cdot \vec{\epsilon}_1 = 0$ $\qquad$ $\hat{n} \cdot \vec{\epsilon}_2 = \sin(\theta_i) = \hat{n} \cdot \vec{\epsilon}_2{}''$ $\qquad$ $\hat{n} \cdot \vec{\epsilon}_2{}' = \sin(\theta_b)$

$\hat{n} \times \vec{\epsilon}_2 = \vec{\epsilon}_1 \cos(\theta_i)$ $\qquad$ $\hat{n} \times \vec{\epsilon}_2{}' = \vec{\epsilon}_1 \cos(\theta_b)$ $\qquad$ $\hat{n} \times \vec{\epsilon}_2{}'' = -\hat{\epsilon}_1 \cos(\theta_i)$

$\vec{\epsilon}_1$ is chosen to be perpendicular to the plane of incidence. Decompose the electric fields of the waves:

$$\vec{E}_0 = E_{01}\, \vec{\epsilon}_1 + E_{02}\, \vec{\epsilon}_2 \qquad \text{where e.g. } E_{01} = \vec{E}_0 \cdot \vec{\epsilon}_1$$

$$\vec{E}_0{}' = E_{01}'\, \vec{\epsilon}_1{}' + E_{02}'\, \vec{\epsilon}_2{}'$$

$$\vec{E}_0{}'' = E_{01}''\, \vec{\epsilon}_1{}'' + E_{02}''\, \vec{\epsilon}_2{}''$$

Back to the boundary conditions at the interface:

$$\hat{n} \cdot \Delta\vec{D} = 0 \implies \boxed{\epsilon\left(E_{02} + E_{02}''\right)\sin(\theta_i) - \epsilon' E_{02}' \sin(\theta_b) = 0}$$

$$\hat{n} \times \Delta\vec{E} = 0 \implies \left(E_{01} + E_{01}'' - E_{01}'\right)\left(\vec{\epsilon}_1 \times \hat{n}\right)$$

$$- \left[\left(E_{02} - E_{02}''\right)\cos(\theta_i) - E_{02}' \cos(\theta_b)\right] \vec{\epsilon}_1 = 0$$

This is a <u>vector</u> equation and both perpendicular components are zero separately. That is

$$\boxed{E_{01} + E_{01}'' - E_{01}' = 0}$$

and

$$\boxed{\left(E_{02} - E_{02}''\right)\cos(\theta_i) - E_{02}' \cos(\theta_b) = 0}$$

$$\hat{n}\cdot\Delta\vec{B}=0 \Rightarrow \left(k E_{01} + k'' E_{01}''\right)\sin(\theta_i) - k' E_{01}'\sin(\theta_b) = 0$$

using $k = \frac{\omega n}{c} = k''$ and $k' = \frac{\omega n'}{c}$

with Snell's Law $n\sin(\theta_i) = n'\sin(\theta_b)$

This equation becomes

$$E_{01} + E_{01}'' - E_{01}' = 0$$

but we knew this already from the previous page, so nothing new here.

$$\hat{n}\times\Delta\vec{H}=0 \Rightarrow -\left[\frac{1}{\mu}\left(k E_{01} - k'' E_{01}''\right)\cos(\theta_i) - \frac{1}{\mu'} k' E_{01}'\cos(\theta_b)\right]\vec{\epsilon}_1$$

$$+\left[\frac{1}{\mu}\left(k E_{02} + k'' E_{02}''\right) - \frac{1}{\mu'} k' E_{02}'\right]\vec{\epsilon}_1\times\hat{n} = 0$$

this is another <u>vector</u> equation so both terms = 0.

$$\boxed{\frac{n}{\mu}\left(E_{01} - E_{01}''\right)\cos(\theta_i) - \frac{n'}{\mu'} E_{01}'\cos(\theta_b) = 0}$$

and

$$\boxed{\frac{n}{\mu}\left(E_{02} + E_{02}''\right) - \frac{n'}{\mu'} E_{02}' = 0}$$

Now we have 5 equations (in the boxes) and only 4 unknowns $\left\{\frac{E_{01}'}{E_{01}}, \frac{E_{01}''}{E_{01}}, \frac{E_{02}'}{E_{02}}, \frac{E_{02}''}{E_{02}}\right\}$ so the equations can not be linearly independent. (The incident amplitudes E_{01} and E_{02} are arbitrary, so ratios are the unknowns).