$$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = -i\, 2\pi^2 I_0 \tau_0 e^{-|\omega|\tau_0} a\, \hat{y}\, J_1(ka \sin\theta_k)$$

$$W(\omega, \Omega) = \frac{\mu_0 \omega^2}{16\pi^3 c}\left[\tilde{\tilde{\vec{J}}}^*_{\vec{k}\omega} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega} - \left(\hat{n}\cdot\tilde{\tilde{\vec{J}}}^*_{\vec{k}\omega}\right)\left(\hat{n}\cdot\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)\right]$$

Note that because we chose $\vec{k}$ in the xz-plane,

$\hat{n}\cdot\hat{y} = 0$. So

$$W(\omega, \Omega) = \frac{\mu_0 \omega^2}{16\pi^3 c} \tilde{\tilde{\vec{J}}}^*_{\vec{k}\omega} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega}$$

$$= \frac{\mu_0 \omega^2}{16\pi^3 c} 4\pi^4 I_0^2 \tau_0^2 e^{-2|\omega|\tau_0} a^2\, J_1^2\left(\frac{\omega a \sin\theta_k}{c}\right)$$

For $\lambda \gg a$, then $\omega a = \frac{2\pi a}{\lambda} \ll 1$ and

$J_1^2\left(\frac{2\pi a \sin\theta_k}{\lambda}\right)$ can be expanded in a

power series in $\left(\frac{a}{\lambda}\right)$.

The other usual case is a <u>periodic</u> system for which we have a Fourier series rather than a Fourier transform.

$$\mathbb{X}(\vec{r},t) = \sum_{n=-\infty}^{\infty} \tilde{\mathbb{X}}_{\omega_n}(\vec{r})\, e^{-i\omega_n t}$$

where the period is $T = \frac{2\pi}{\omega}$ and ω is the fundamental frequency (lowest frequency).

Then $\omega_n = n\omega = \frac{2\pi n}{T}$ are the harmonics,

In all the previous analysis, simply replace ω by ω_n.

For $r \to \infty$, we have

$$\tilde{\vec{A}}_{\omega_n}(\vec{r}) \longrightarrow \frac{\mu_0}{4\pi} \frac{e^{i\frac{\omega_n}{c}r}}{r} \left[\tilde{\tilde{\vec{J}}}_{\vec{k}_n \omega_n}\right]_{tr}$$

$$\tilde{\vec{E}}_{\omega_n}(\vec{r}) \longrightarrow i\omega_n \frac{\mu_0}{4\pi} \frac{e^{\frac{i\omega_n r}{c}}}{r} \left[\tilde{\tilde{\vec{J}}}_{\vec{k}_n \omega_n}\right]_{tr}$$

$$\tilde{\vec{B}}_{\omega_n}(\vec{r}) \longrightarrow \frac{i\omega_n}{c} \frac{\mu_0}{4\pi} \frac{e^{\frac{i\omega_n r}{c}}}{r}\, \hat{n} \times \left[\tilde{\tilde{\vec{J}}}_{\vec{k}_n \omega_n}\right]_{tr} = \frac{\hat{n}}{c} \times \tilde{\vec{E}}_{\omega_n}(\vec{r})$$

where $\vec{k}_n = \frac{\omega_n}{c}\hat{n}$ and $\hat{n} = \frac{\vec{r}}{r}$

the instantaneous <u>power</u> transmitted through a sphere of radius r is

$$P(t) = \iint d\Omega \, r^2 \hat{n} \cdot \vec{\mathcal{S}}(\vec{r},t) = \frac{1}{\mu_0}\iint d\Omega \, r^2 \hat{n} \cdot \vec{\mathcal{E}}(\vec{r},t) \times \vec{\mathcal{B}}(\vec{r},t)$$

$$= \frac{1}{\mu_0}\iint d\Omega \, r^2 \hat{n} \cdot \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{\vec{E}}_{\omega_n}(\vec{r}) \times \tilde{\vec{B}}_{\omega_m}(\vec{r}) \, e^{-i(\omega_n+\omega_m)t}$$

Now average the power over one period T

$$\langle P \rangle = \frac{1}{T}\int_0^T dt \, P(t) = \frac{1}{\mu_0}\iint d\Omega \, r^2 \hat{n} \cdot \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{\vec{E}}_{\omega_n}(\vec{r}) \times \tilde{\vec{B}}_{\omega_m}(\vec{r}) \, \delta_{n(-m)}$$

$$= \frac{1}{\mu_0} \sum_{n=-\infty}^{\infty} \iint d\Omega \, r^2 \hat{n} \cdot \tilde{\vec{E}}_{\omega_n}(\vec{r}) \times \tilde{\vec{B}}^{*}_{\omega_n}(\vec{r})$$

where we used $\tilde{\vec{B}}_{\omega_{-n}}(\vec{r}) = \tilde{\vec{B}}_{-\omega_n}(\vec{r}) = \tilde{\vec{B}}^{*}_{\omega_n}(\vec{r})$

For $r \to \infty$, we find

$$\langle P \rangle = \frac{1}{\mu_0 c} \sum_{n=-\infty}^{\infty} \left(\frac{\mu_0}{4\pi}\right)^2 \omega_n^2 \iint d\Omega \left[\tilde{\tilde{\vec{J}}}^{*}_{\vec{k}_n \omega_n} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}_n \omega_n} - \left(\hat{n} \cdot \tilde{\tilde{\vec{J}}}^{*}_{\vec{k}_n \omega_n}\right)\left(\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}_n \omega_n}\right) \right]$$

Multipole Expansions

There are 3 length scales in our analysis:

(i) The "size" of the source : $\underline{r'}$ or $\underline{a}$

(ii) The distance to the field point : $\underline{r}$

(iii) The wavelength: $\underline{\lambda}$

When $r > a$ we can make multipole expansions analogous to what we did in electro- and magnetostatics. Here we are assuming that $r \gg a$ and hence only radiation fields are contributing. We also have $r \gg \lambda$. But in general we have an arbitrary ratio $\frac{r'}{\lambda}$. In general the space-time transformed current density $\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}$ is a complicated function, but for $\lambda \gg r'$ we can simplify matters a bit by expanding $e^{-i\vec{k}\cdot\vec{r}'} = 1 - i\vec{k}\cdot\vec{r}' - \frac{1}{2}(\vec{k}\cdot\vec{r}')^2 - + \cdots$

and so for the m^{th} component of $\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}$ we have

$$\left(\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)_m = \iiint \left(\tilde{\vec{J}}_\omega(\vec{r}')\right)_m dV' - i\sum_{p=1}^{3} k_p \iiint x_p' \left(\tilde{\vec{J}}_\omega(\vec{r}')\right)_m dV'$$

$$- \frac{1}{2}\sum_{n,p=1}^{3} k_n k_p \iiint x_n' x_p' \left(\tilde{\vec{J}}_\omega(\vec{r}')\right)_m dV'$$

Thus we need the various <u>moments</u> of the time Fourier transformed current density $\tilde{\vec{J}}_\omega(\vec{r}')$. These can be constructed from various numerical tensors (like δ_{ij} and ϵ_{ijk}) and tensors that define the charge and current distributions:

electric dipole moment: $\iiint x_i' \, \tilde{\rho}_\omega(\vec{r}') \, dV' \equiv (\tilde{\vec{P}}_\omega)_i$ or $(\tilde{\vec{Q}}_\omega)_i$

magnetic dipole moment: $\frac{1}{2}\iiint \left[\vec{r}' \times \tilde{\vec{J}}_\omega(\vec{r}')\right]_i dV' \equiv (\tilde{\vec{m}}_\omega)_i$

electric quadrupole moment: $\iiint x_i' \, x_j' \, \tilde{\rho}_\omega(\vec{r}') \, dV' \equiv (\tilde{Q}_\omega)_{ij}$

magnetic quadrupole moment:

$$\frac{1}{3}\iiint \left\{ x_i' \left[\vec{r}' \times \tilde{\vec{J}}_\omega(\vec{r}')\right]_j + x_j' \left[\vec{r}' \times \tilde{\vec{J}}_\omega(\vec{r}')\right]_i \right\} dV' \equiv (\tilde{m}_\omega)_{ij}$$

etc.

For a localized current density, it follows from the divergence theorem that

$$\iiint \vec{\nabla}' \cdot \left[x'_m \, \vec{\tilde{J}}_\omega(\vec{r}\,') \right] dV' = \oiint_{S_\infty} \hat{n}' \cdot \vec{\tilde{J}}_\omega(\vec{r}\,') \, x'_m = 0$$

since $\vec{\tilde{J}}_\omega(\vec{r}\,')$ vanishes outside a sphere of sufficiently large radius. But

$$\iiint \vec{\nabla}' \cdot \left[x'_m \, \vec{\tilde{J}}_\omega(\vec{r}\,') \right] dV' = \iiint \left[x'_m \left[\vec{\nabla}' \cdot \vec{\tilde{J}}_\omega(\vec{r}\,') \right] + \vec{\tilde{J}}_\omega(\vec{r}\,') \cdot \underbrace{\left(\vec{\nabla}' x'_m \right)}_{\hat{e}_m} \right] dV' = 0$$

therefore

$$\iiint \left[\vec{\tilde{J}}_\omega(\vec{r}\,') \right]_m dV' = - \iiint x'_m \left[\vec{\nabla}' \cdot \vec{\tilde{J}}_\omega(\vec{r}\,') \right] dV'$$

From the equation of continuity: $\vec{\nabla}' \cdot \vec{\tilde{J}}_\omega(\vec{r}\,') = i\omega \tilde{\rho}_\omega(\vec{r}\,')$

$$\iiint \left[\vec{\tilde{J}}_\omega(\vec{r}\,') \right]_m dV' = -i\omega \iiint x'_m \, \tilde{\rho}_\omega(\vec{r}\,') \, dV' = -i\omega \left(\vec{\tilde{p}}_\omega \right)_m$$

So

$$\boxed{\iiint \vec{\tilde{J}}_\omega(\vec{r}\,') \, dV' = -i\omega \, \vec{\tilde{p}}_\omega} = -i\omega \, \vec{\tilde{Q}}_\omega$$

electric dipole

Next, we write

$$\iiint x_i' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_m dV' = \frac{1}{2}\iiint \left\{ x_i' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_m - x_m' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_i \right\} dV' +$$

$$+ \frac{1}{2}\iiint \left\{ x_i' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_m + x_m' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_i \right\} dV'$$

but the antisymmetric piece is

$$\frac{1}{2}\iiint \left\{ x_i' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_m - x_m' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_i \right\} dV' = \sum_\ell \epsilon_{im\ell} \left(\vec{\tilde{m}}_\omega\right)_\ell$$

To determine the symmetric piece, note that

$$\iiint \vec{\nabla}' \cdot \left[x_i' x_m' \vec{\tilde{J}}_\omega(\vec{r}\,')\right] dV' = 0 \quad \text{by the divergence theorem}$$

$$0 = \iiint \left\{ x_i' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_m + x_m' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_i + x_i' x_m' \vec{\nabla}' \cdot \vec{\tilde{J}}_\omega(\vec{r}\,') \right\} dV'$$

So $\frac{1}{2}\iiint \left\{ x_i' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_m + x_m' \left[\vec{\tilde{J}}_\omega(\vec{r}\,')\right]_i \right\} dV' =$

$$= -\frac{1}{2}\iiint x_i' x_m' \, i\omega \, \tilde{\rho}_\omega(\vec{r}\,') dV'$$

$$= \frac{-i\omega}{2} \left(\tilde{Q}_\omega\right)_{im}$$

Putting everything together

$$\iiint x_i' \left[\tilde{\vec{J}}_\omega(\vec{r}')\right]_m dV' = \sum_\ell{}' \epsilon_{im\ell} \left(\tilde{\vec{m}}_\omega\right)_\ell - \frac{i\omega}{2}\left(\tilde{Q}_\omega\right)_{im}$$

$\uparrow$ magnetic dipole $\qquad$ $\uparrow$ electric quadrupole

We are now in a position to determine the contribution to the electric and magnetic fields from the various multipoles. We have from the bottom of page 3:

$$\left(\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)_m = -i\omega\left(\tilde{\vec{p}}_\omega\right)_m - i\sum_i\sum_\ell k_i\, \epsilon_{i\ell m}\left(\tilde{\vec{m}}_\omega\right)_m$$

$$-i\left(\frac{-i\omega}{2}\right)\sum_i k_i\left(\tilde{Q}_\omega\right)_{im} + \cdots$$

or in vector notation

$$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = -i\omega\,\tilde{\vec{p}}_\omega - i\frac{\omega}{c}\,\tilde{\vec{m}}_\omega \times \hat{n} - \frac{1}{2}\frac{\omega^2}{c}\,\tilde{\overleftrightarrow{Q}}_\omega \cdot \hat{n} + \cdots$$

$\uparrow$ dyad

Next, we look at the fields for the individual multipoles. Recall $\tilde{\vec{A}}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\left[\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right]_{tr}$

Recall that for the radiation fields

$$\vec{\tilde{B}}_\omega(\vec{r}) = \vec{\nabla} \times \vec{\tilde{A}}_\omega(\vec{r}) = \frac{i\omega}{c}\,\hat{n} \times \vec{\tilde{A}}_\omega(\vec{r}) + \text{non-radiative terms}$$

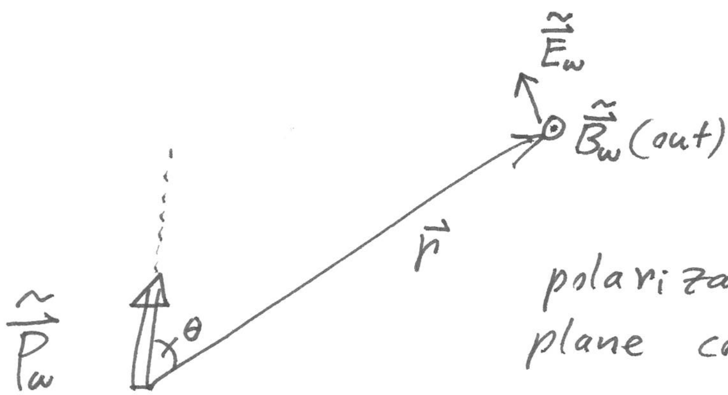
$$\vec{\tilde{E}}_\omega(\vec{r}) = c\,\vec{\tilde{B}}_\omega(\vec{r}) \times \hat{n} + \text{non-radiative terms}$$

E1 (electric dipole)

$$\vec{\tilde{B}}_\omega(\vec{r}) = \frac{i\omega}{c}\,\hat{n} \times \frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\left(-i\omega\vec{\tilde{P}}_\omega\right)$$

$$= \frac{e^{i\frac{\omega}{c}r}}{r}\frac{\mu_0}{4\pi}\frac{\omega^2}{c}\,\hat{n} \times \vec{\tilde{P}}_\omega$$

$$\vec{\tilde{E}}_\omega(\vec{r}) = \frac{e^{i\frac{\omega}{c}r}}{r}\frac{\mu_0}{4\pi}\omega^2\left(\hat{n} \times \vec{\tilde{P}}_\omega\right) \times \hat{n}$$



polarization is in the plane containing $\vec{r}$ and $\vec{\tilde{P}}_\omega$.

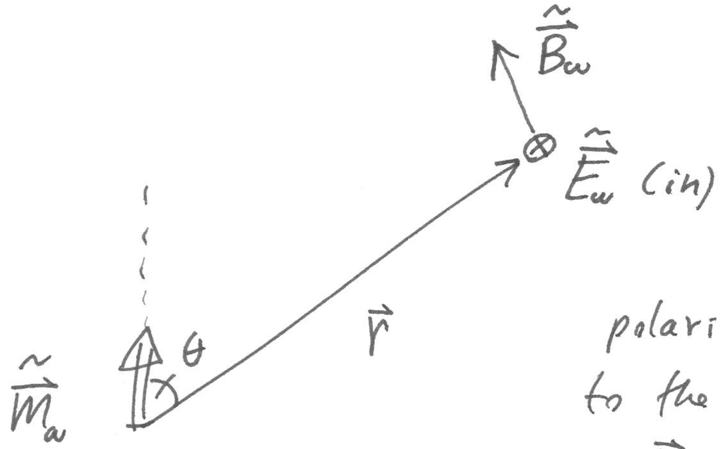
M1 (magnetic dipole)

$$\tilde{\vec{B}}_\omega(\vec{r}) = \frac{i\omega}{c}\hat{n}\times\frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\left(-i\frac{\omega}{c}\tilde{\vec{m}}_\omega\times\hat{n}\right)$$

$$= \frac{e^{i\frac{\omega}{c}r}}{r}\frac{\mu_0}{4\pi}\frac{\omega^2}{c^2}\hat{n}\times(\tilde{\vec{m}}_\omega\times\hat{n})$$

$$\tilde{\vec{E}}_\omega(\vec{r}) = \frac{e^{i\frac{\omega}{c}r}}{r}\frac{\mu_0}{4\pi}\frac{\omega^2}{c}\left[\hat{n}\times(\tilde{\vec{m}}_\omega\times\hat{n})\right]\times\hat{n}$$

$$= \frac{e^{i\frac{\omega}{c}r}}{r}\frac{\mu_0}{4\pi}\frac{\omega^2}{c}\tilde{\vec{m}}_\omega\times\hat{n}$$



polarization is perpendicular to the plane containing $\vec{r}$ and $\tilde{\vec{m}}_\omega$.

E2 (electric quadrupole)

$$\tilde{\vec{B}}_\omega(\vec{r}) = \frac{i\omega}{c}\,\hat{n} \times \frac{\mu_0}{4\pi}\,\frac{e^{i\frac{\omega}{c}r}}{r}\left(-\frac{1}{2}\frac{\omega^2}{c}\,\tilde{\overleftrightarrow{Q}}_\omega \cdot \hat{n}\right)$$

$$= \frac{-i\omega^3}{2c^2}\,\frac{e^{i\frac{\omega}{c}r}}{r}\,\frac{\mu_0}{4\pi}\,\hat{n} \times \left(\tilde{\overleftrightarrow{Q}}_\omega \cdot \hat{n}\right)$$

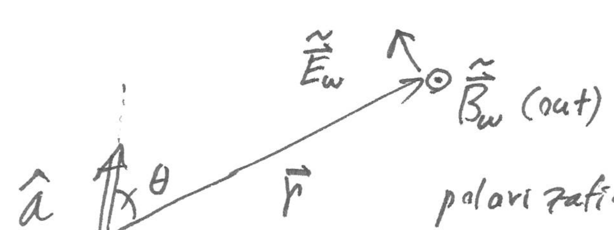
$$\tilde{\vec{E}}_\omega(\vec{r}) = -\frac{i\omega^3}{2c}\,\frac{e^{i\frac{\omega}{c}r}}{r}\,\frac{\mu_0}{4\pi}\left[\hat{n} \times \left(\tilde{\overleftrightarrow{Q}}_\omega \cdot \hat{n}\right)\right] \times \hat{n}$$

For the special case of a <u>linear</u> <u>quadrupole</u>

$$\tilde{\overleftrightarrow{Q}}_\omega = f_\omega\,\hat{a}\,\hat{a} \qquad \text{exterior product}$$

$$\tilde{\vec{B}}_\omega(\vec{r}) = \frac{-i\omega^3}{2c^2}\,\frac{e^{i\frac{\omega}{c}r}}{r}\,\frac{\mu_0}{4\pi}\,f_\omega\,(\hat{n} \times \hat{a})(\hat{n} \cdot \hat{a})$$

$$\tilde{\vec{E}}_\omega(\vec{r}) = \frac{-i\omega^3}{2c}\,\frac{e^{i\frac{\omega}{c}r}}{r}\,\frac{\mu_0}{4\pi}\,f_\omega\left[(\hat{n} \times \hat{a}) \times \hat{n}\right](\hat{n} \cdot \hat{a})$$



polarization is in the plane containing $\hat{a}$ and $\vec{r}$