We may also consider the power radiated. For example, consider multipoles that have only one Fourier component. Then the expression for the average power radiated by a periodic system is (with $n = \pm 1$ only)

$$\langle P \rangle = \iint d\Omega \, \frac{dP}{d\Omega}$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 \omega^2}{8\pi^2 c} \left[\tilde{\tilde{\vec{J}}}_{k\omega}^{\,*} \cdot \tilde{\tilde{\vec{J}}}_{k\omega} - \left(\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{k\omega}^{\,*}\right)\left(\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{k\omega}\right) \right]$$

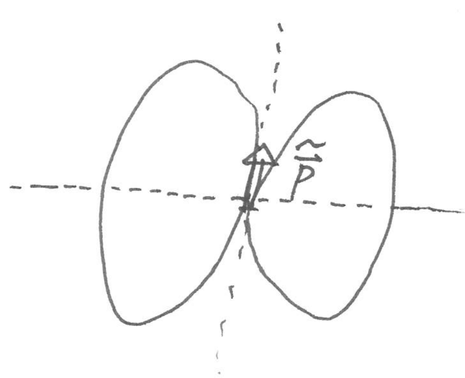
where $\tilde{\tilde{\vec{J}}}_{k\omega} = \tilde{\tilde{\vec{J}}}_{k_n \omega_n}$ for $\omega_n = \omega_1 = \omega$, $\vec{k}_n = \vec{k}_1 = \frac{\omega}{c}\hat{n}$

For the various multipoles, we have

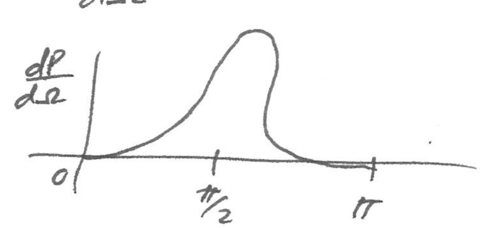
E1 $\quad \dfrac{dP}{d\Omega} = \dfrac{\mu_0 \omega^4}{8\pi^2 c} \left[\tilde{\vec{P}}_\omega^{\,*} \cdot \tilde{\vec{P}}_\omega - \left(\hat{n} \cdot \tilde{\vec{P}}_\omega^{\,*}\right)\left(\hat{n} \cdot \tilde{\vec{P}}_\omega\right) \right]$

For $\tilde{\vec{P}}_\omega = \tilde{P}_\omega \hat{a}$, we have

$$\frac{dP}{d\Omega} = \frac{\mu_0 \omega^4}{8\pi^2 c} \left|\tilde{P}_\omega\right|^2 \sin^2\theta$$



Parametric plot of $\frac{dP}{d\Omega}$ vs. θ



M1

$$\frac{dP}{d\Omega} = \frac{\mu_0 \omega^4}{8\pi^2 c^3}\left[\tilde{\vec{m}}_\omega^* \cdot \tilde{\vec{m}}_\omega - (\hat{n}\cdot\tilde{\vec{m}}_\omega^*)(\hat{n}\cdot\tilde{\vec{m}}_\omega)\right]$$

For $\tilde{\vec{m}}_\omega = \tilde{m}_\omega \hat{a}$ we have

$$\frac{dP}{d\Omega} = \frac{\mu_0 \omega^4}{8\pi^2 c^3}|\tilde{m}_\omega|^2 \sin^2\theta$$

the pattern resembles that for E1.

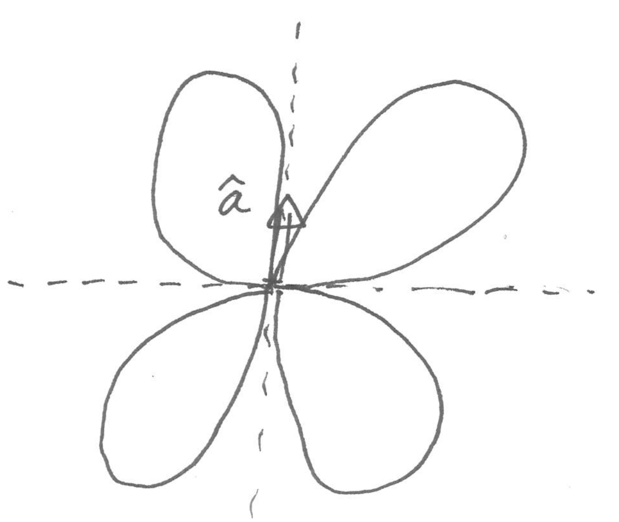
E2

$$\frac{dP}{d\Omega} = \frac{\mu_0 \omega^6}{8\pi^2 c^3}\frac{1}{4}\left|\hat{n}\times(\tilde{\overleftrightarrow{Q}}_\omega\cdot\hat{n})\right|^2$$

for $\tilde{\overleftrightarrow{Q}}_\omega = \tilde{f}_\omega \hat{a}\hat{a}$ we have

$$\frac{dP}{d\Omega} = \frac{\mu_0 \omega^6}{8\pi^2 c^3}\frac{1}{4}|\tilde{f}_\omega|^2 \sin^2\theta\cos^2\theta$$

with radiation pattern



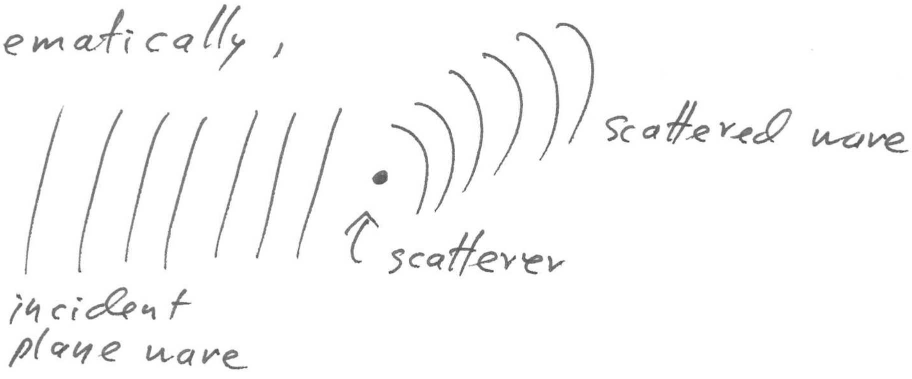
For multipoles of order L, the power radiated goes like $(\omega a)^{2L}$ where $\underline{a}$ is the "size" of the source. [The multipole of order $L \sim q a^L$].

The higher the order of the multipole, the more nodes in the radiation pattern. For a <u>linear</u> multipole tensor $(T\underbrace{\hat{a}\hat{a}\ldots\hat{a}}_{L \text{ factors}})$, the number of nodes found in the plot (including the $\hat{a}$-axis) is $(L+1)$.

Scattering

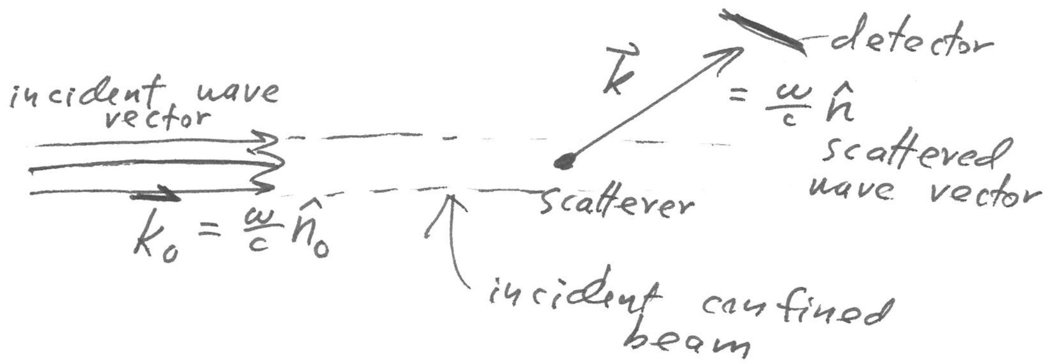
The basic scattering process consists of the following. A plane wave is incident on a scatterer (e.g. a molecule). The charges making up the scatterer (primarily electrons) respond to the electric field of the incident wave (the magnetic forces are of order $\frac{v}{c}$ and hence much weaker than the electric force). The driven motion of the scatterer will produce radiation — <u>scattered radiation</u>.

Schematically,



If the object upon which the incident wave impinges consists of many molecular scatterers, then the scattered wave themselves become scattered and the physics can be very complicated. In certain cases there are simplifications (for example, wave propagation in a homogeneous medium made up of microscopic "scatterers" was quite straightforward).

We will restrict the discussion to a plane wave (wave vector $\vec{k}_0 = \frac{\omega}{c}\hat{n}_0$) incident on a scatterer of size $\underline{a}$, with $\frac{\omega a}{c} \ll 1$. In reality, the wave will <u>not</u> be an infinite plane wave but will be "confined" in directions perpendicular to $\hat{n}_0$. Hence, we must mix in wavevectors to build this transverse wave packet and the wave will not be exactly monochromatic. But we assume that this frequency spread is not significant and that the transverse wave confinement is sufficient to avoid interference between the scattered and incident radiation in directions other than forward.



Thus the detector is outside the incident beam except for $\vec{k} \simeq \vec{k}_0$.

The formalism that describes scattering is essentially that developed to describe radiation. We want outgoing waves from the scatterer together with the incident plane wave $\sim e^{i\vec{k}_0\cdot\vec{r}}$. We take the solution of the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = -\mu_0 \vec{J}_{tr}(\vec{r},t)$$

to be the monochromatic wave

$$\vec{A}(\vec{r},t) = \tilde{\vec{A}}_\omega(\vec{r})\, e^{-i\omega t} + \tilde{\vec{A}}_\omega^{\,*}(\vec{r})\, e^{i\omega t}$$

$$\tilde{\vec{A}}_\omega(\vec{r}) = \tilde{\vec{A}}_\omega^{(in)} e^{i\vec{k}_0\cdot\vec{r}} + \frac{\mu_0}{4\pi}\iiint dV' \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\left[\tilde{\vec{J}}_\omega(\vec{r}')\right]_{tr}$$

$$\xrightarrow[r\to\infty]{} \tilde{\vec{A}}_\omega^{(in)} e^{i\vec{k}_0\cdot\vec{r}} + \frac{\mu_0}{4\pi}\frac{e^{i\vec{k}_0\cdot\vec{r}}}{r}\left[\tilde{\hat{\vec{J}}}_{\vec{k}\omega} - \hat{n}\left(\hat{n}\cdot\tilde{\hat{\vec{J}}}_{\vec{k}\omega}\right)\right]$$

The current accurring in this expression is induced by the incident electric field. Because we are assuming that $\frac{\omega a}{c} \ll 1$, we limit the discussion to electric and magnetic dipole radiation. The electric field from the vector potential above is

$$\vec{E}(\vec{r},t) = \tilde{\vec{E}}_\omega(\vec{r})\, e^{-i\omega t} + \tilde{\vec{E}}_\omega^{*}(\vec{r})\, e^{i\omega t}$$

$$\tilde{\hat{\vec{E}}}_\omega(\vec{r}) = \tilde{\hat{\vec{E}}}_\omega^{(in)} e^{i\vec{k}_0\cdot\vec{r}} + \frac{\mu_0}{4\pi}(i\omega)\frac{e^{i\frac{\omega}{c}r}}{r}\left(\hat{n}\times\tilde{\hat{\vec{J}}}_{\vec{k}\omega}\right)\times\hat{n}$$

$$\tilde{\hat{\vec{J}}}_{\vec{k}\omega} = -i\omega\left[\tilde{\vec{P}}_\omega + \frac{1}{c}\tilde{\vec{m}}_\omega\times\hat{n}\right]$$

and we will use: $[\hat{n}\times(\tilde{\vec{m}}_\omega\times\hat{n})]\times\hat{n} = \tilde{\vec{m}}_\omega\times\hat{n}$.

$$\tilde{\hat{\vec{E}}}_\omega(\vec{r}) = \tilde{\hat{\vec{E}}}_\omega^{(in)} e^{i\vec{k}_0\cdot\vec{r}} + \frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\omega^2\left[(\hat{n}\times\tilde{\vec{P}}_\omega)\times\hat{n} + \frac{1}{c}\tilde{\vec{m}}_\omega\times\hat{n}\right]$$

Because the dipole moments are induced we assume that

$$\tilde{\vec{p}}_\omega = \alpha \tilde{\vec{E}}_\omega^{(in)} \qquad \text{and} \qquad \tilde{\vec{m}}_\omega = \bar{\alpha} \tilde{\vec{B}}_\omega^{(in)} = \bar{\alpha}\, \frac{\hat{n}_0}{c} \times \tilde{\vec{E}}_\omega^{(in)}$$

Thus

$$\tilde{\vec{E}}_\omega(\vec{r}) = \tilde{\vec{E}}_\omega^{(in)} e^{i\vec{k}_0\cdot\vec{r}} + \tilde{\vec{E}}_\omega^{(sc)}(\vec{r}) \qquad \text{where}$$

$$\tilde{\vec{E}}_\omega^{(sc)}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\frac{\omega}{c}r}}{r} \omega^2 \left[\alpha (\hat{n} \times \tilde{\vec{E}}_\omega^{(in)}) \times \hat{n} + \frac{\bar{\alpha}}{c^2} (\hat{n}_0 \times \tilde{\vec{E}}_\omega^{(in)}) \times \hat{n} \right]$$

For simplicity, we assume from now on that we have only electric dipole radiation. Then if the incident wave is linearly polarized, the scattered wave is also linearly polarized in the plane containing $\tilde{\vec{E}}_\omega^{(in)}$ and $\hat{n}$.

In general, we expand $\tilde{\vec{E}}_\omega^{(in)}$ in terms of arbitrary polarization vectors $\hat{\epsilon}_j^{(in)}$ $(j=1,2)$

$$\tilde{\vec{E}}_\omega^{(in)} = \sum_j \hat{\epsilon}_j^{(in)} \tilde{E}_{\omega j}^{(in)}$$

In a similar manner we expand

$$\tilde{\vec{E}}_\omega^{(sc)}(\vec{r}) = \sum_i \hat{\epsilon}_i^{(sc)} \tilde{E}_{\omega j}^{(sc)}(\vec{r}) .$$

These polarization vectors, which may be complex, satisfy:

$$\hat{\epsilon}_i^{(in)*} \cdot \hat{\epsilon}_j^{(in)} = \delta_{ij} \qquad \hat{\epsilon}_i^{(in)} \cdot \hat{n}_o = 0$$

$$\hat{\epsilon}_i^{(sc)*} \cdot \hat{\epsilon}_j^{(sc)} = \delta_{ij} \qquad \hat{\epsilon}_i^{(sc)} \cdot \hat{n} = 0$$

We define the 2×2 "scattering matrix" $\underline{\underline{M}}$ through

$$\tilde{E}^{(sc)}_{\omega i}(\vec{r}) = \frac{e^{i\frac{\omega}{c}r}}{r} \sum_j M_{ij} E^{(in)}_{\omega j}$$

$$M_{ij} = \frac{\mu_0}{4\pi} \omega^2 \alpha \left[\hat{\epsilon}_i^{(sc)*} \cdot \hat{\epsilon}_j^{(in)} - \underbrace{(\hat{n} \cdot \hat{\epsilon}_i^{(sc)*})}_{0}(\hat{n} \cdot \hat{\epsilon}_j^{(in)}) \right]$$

$$M_{ij} = \frac{\mu_0}{4\pi} \omega^2 \alpha \, \hat{\epsilon}_i^{(sc)*} \cdot \hat{\epsilon}_j^{(in)}$$

To analyze the scattering, we calculate the time-averaged power for radiation scattered into a collection of detectors distributed over all solid angle at infinity. We ignore the interference problem in the forward direction — if $r \to \infty$ and the incident beam has transverse confinement, then only the forward direction will have interference and this is a negligible contribution to the solid angle. Thus

$$\langle P \rangle = \oiint_{S_\infty} dS \; \hat{n} \cdot \langle \vec{\mathcal{S}}_{scat}(\vec{r},t) \rangle = \iint d\Omega \, r^2 \frac{2}{\mu_0} \tilde{\vec{E}}_\omega^{(sc)*}(\vec{r}) \cdot \tilde{\vec{E}}_\omega^{(sc)}(\vec{r})$$

$$\equiv \iint d\Omega \frac{dP}{d\Omega}$$ where the differential scattered power is

$$\frac{dP}{d\Omega} = \frac{2}{\mu_0} \sum_i \sum_j \sum_k M_{ij} \tilde{E}_{\omega j}^{(in)} M_{ik}^* \tilde{E}_{\omega k}^{(in)*}$$

$$= \frac{2}{\mu_0} \mathrm{Tr} \left[\underline{\underline{M}} \, \underline{\underline{E}}_\omega^{(in)} \underline{\underline{E}}_\omega^{(in)\dagger} \underline{\underline{M}}^\dagger \right]$$

The energy per unit time per unit solid angle is proportional to the square of the incident amplitude $\tilde{\vec{E}}_\omega^{(in)*} \cdot \tilde{\vec{E}}_\omega^{(in)}$

In fact, the energy per unit area per unit time incident on the target is given by

$$I_{in} = \hat{n}_0 \cdot \langle \vec{\mathcal{S}}_{in}(\vec{r},t) \rangle = \frac{2}{\mu_0} \tilde{\vec{E}}_\omega^{(in)*} \cdot \tilde{\vec{E}}_\omega^{(in)}$$

So we write $$\frac{dP}{d\Omega} = I_{in}\,\sigma(\Omega)$$

where $\sigma(\Omega)$ is the differential scattering cross section.

It is independent of $|\tilde{\vec{E}}_\omega^{(in)}|$ but does depend on the direction of $\hat{\vec{E}}_\omega^{(in)}$, that is on the polarization.

$\sigma(\Omega)$ has the dimensions of area and $\sigma(\Omega)\,d\Omega$ is the effective area for removing incident energy from the beam and scattering it into solid angle $d\Omega$ about the direction Ω, that is θ and φ which are the "scattering angles" which define the direction $\hat{n}$.

From the definitions then

$$\sigma(\Omega) = \frac{Tr\left[\underline{\underline{M}}\;\underline{\underline{E}}_\omega^{(in)}\underline{\underline{E}}_\omega^{(in)\dagger}\underline{\underline{M}}^\dagger\right]}{Tr\left[\underline{\underline{E}}_\omega^{(in)}\underline{\underline{E}}_\omega^{(in)\dagger}\right]}$$

and $dP = I_{in}\,\sigma(\Omega)\,d\Omega$