In the usual case that the radiation is only partially polarized, we replace $\tilde{\vec{E}}_\omega^{(in)} \tilde{\vec{E}}_\omega^{(in)\dagger}$ by a matrix $\underline{\underline{J}}^{(in)} = \underline{\underline{J}}^{(in)\dagger}$ as discussed at the beginning of the course.

In this more general case, we have

$$\sigma(\Omega) = \text{Tr}\left[\underline{\underline{M}} \, \underline{\underline{\rho}}^{in} \, \underline{\underline{M}}^\dagger\right]$$

where the density matrix for the incident wave is

$$\underline{\underline{\rho}}_{in} = \frac{\underline{\underline{J}}^{(in)}}{\text{Tr}\left[\underline{\underline{J}}^{(in)}\right]}$$

From the linear relation between $\tilde{\vec{E}}_\omega^{(in)}$ and $\tilde{\vec{E}}_\omega^{(sc)}$ we can also find the density matrix for the scattered radiation

$$\underline{\underline{\rho}}_{sc} = \frac{\underline{\underline{M}} \, \underline{\underline{\rho}}^{in} \, \underline{\underline{M}}^\dagger}{\text{Tr}\left[\underline{\underline{M}} \, \underline{\underline{\rho}}^{in} \, \underline{\underline{M}}^\dagger\right]} = \frac{\underline{\underline{M}} \, \underline{\underline{\rho}}^{in} \, \underline{\underline{M}}^\dagger}{\sigma(\Omega)}$$

The states of polarization of the incident and scattered radiation are described by $\underline{\underline{\rho}}^{in}$ and $\underline{\underline{\rho}}^{sc}$ respectively.

As an example, suppose that the incident beam is unpolarized: $\underline{\underline{\rho}}^{in} = \frac{1}{2}\underline{\underline{\mathbb{I}}}$.

We will consider here only electric dipole scattering for which

$$M_{ij} = \frac{\mu_0}{4\pi} \omega^2 \alpha \, \hat{\epsilon}_i^{(sc)*} \cdot \hat{\epsilon}_j^{(in)}$$

then

$$\left[\underline{\underline{M}} \, \underline{\underline{g}}^{in} \, \underline{\underline{M}}^\dagger \right]_{ij} = \frac{1}{2} \left(\frac{\mu_0}{4\pi} \right)^2 \omega^4 |\alpha|^2 \sum_{\ell=1}^{2} \left[\hat{\epsilon}_i^{(sc)*} \cdot \hat{\epsilon}_\ell^{(in)} \right] \left[\hat{\epsilon}_\ell^{(in)*} \cdot \hat{\epsilon}_j^{(sc)} \right]$$

Now $\hat{\epsilon}_1^{(in)}$, $\hat{\epsilon}_2^{(in)}$, and $\hat{n}_0$ form a triad and completeness (closure) gives

$$\underbrace{\sum_{\ell=1}^{2} \hat{\epsilon}_\ell^{(in)} \hat{\epsilon}_\ell^{(in)*}}_{\text{exterior product}} = \underset{\underset{\text{unit}}{\underset{\text{dyad}}{\uparrow}}}{\overleftrightarrow{I}} - \underbrace{\hat{n}_0 \hat{n}_0}_{\substack{\text{exterior} \\ \text{product}}}$$

Remember $\hat{n}_0$ is the incident beam direction.

Hence

$$\left[\underline{\underline{M}} \, \underline{\underline{g}}^{in} \, \underline{\underline{M}}^\dagger \right]_{ij} = \frac{1}{2} \left(\frac{\mu_0}{4\pi} \right)^2 \omega^4 |\alpha|^2 \left[\underbrace{\hat{\epsilon}_i^{(sc)*} \cdot \hat{\epsilon}_j^{(sc)}}_{\delta_{ij}} - \left(\hat{\epsilon}_i^{(sc)*} \cdot \hat{n}_0 \right) \left(\hat{\epsilon}_j^{(sc)} \cdot \hat{n}_0 \right) \right]$$

Thus the differential cross section is

$$\sigma(\Omega) = Tr \left[\underline{\underline{M}} \, \underline{\underline{g}}^{in} \, \underline{\underline{M}}^\dagger \right] = \sum_{i=1}^{2} \left[\underline{\underline{M}} \, \underline{\underline{g}}^{in} \, \underline{\underline{M}}^\dagger \right]_{ii}$$

$$= \frac{1}{2} \left(\frac{\mu_0}{4\pi} \right)^2 \omega^4 |\alpha|^2 \left[2 - \hat{n}_0 \cdot \underbrace{\sum_{i=1}^{2} \hat{\epsilon}_i^{(sc)} \hat{\epsilon}_i^{(sc)*}}_{\substack{\text{exterior} \\ \text{product}}} \cdot \hat{n}_0 \right]$$

but completeness gives

$$\sum_{i=1}^{2} \hat{\epsilon}_i^{(sc)} \hat{\epsilon}_i^{(sc)*} = \overleftrightarrow{\mathbb{I}} - \hat{n}\hat{n}$$
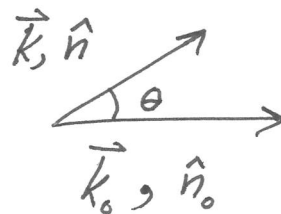
Remember $\hat{n}$ is the scattered beam direction.

There fore

$$\sigma(\Omega) = \frac{1}{2}\left(\frac{\mu_0}{4\pi}\right)^2 \omega^4 |\alpha|^2 \left[2 - \hat{n}_0 \cdot \left(\overleftrightarrow{\mathbb{I}} - \hat{n}\hat{n}\right) \cdot \hat{n}_0\right]$$

$$= \frac{1}{2}\left(\frac{\mu_0}{4\pi}\right)^2 \omega^4 |\alpha|^2 \left[2 - \left(1 - [\hat{n}\cdot\hat{n}_0]^2\right)\right]$$

$$= \frac{1}{2}\left(\frac{\mu_0}{4\pi}\right)^2 \omega^4 |\alpha|^2 \left(1 + \cos^2\theta\right)$$

where θ is the scattering angle.
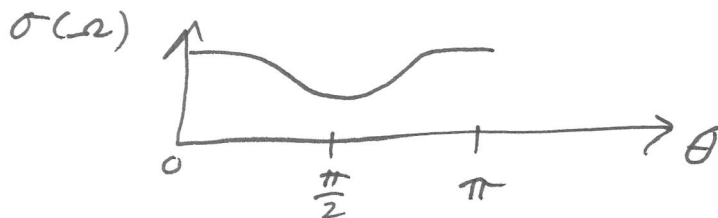

$\vec{k}, \hat{n}$

θ

$\vec{k}_0, \hat{n}_0$

The _total_ cross section is

$$\sigma_{tot} = \iint d\Omega \, \sigma(\Omega) = \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, \sigma(\Omega)$$

$$= \frac{2\pi}{2}\left(\frac{\mu_0}{4\pi}\right)^2 \omega^4 |\alpha|^2 \left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}\left(\frac{\mu_0}{4\pi}\right)^2 \omega^4 |\alpha|^2$$
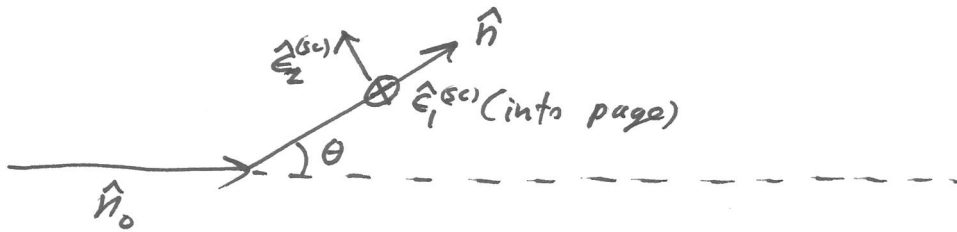
So

$$\sigma(\Omega) = \frac{3}{16\pi} \sigma_{tot} \left(1 + \cos^2\theta\right)$$

$\sigma(\Omega)$

$0 \qquad \frac{\pi}{2} \qquad \pi \qquad \theta$

The scattered density matrix is

$$[\rho^{sc}]_{ij} = \frac{\delta_{ij} - (\hat{\epsilon}_i^{(sc)*} \cdot \hat{n}_0)(\hat{\epsilon}_j^{(sc)} \cdot \hat{n}_0)}{1 + \cos^2\theta}$$
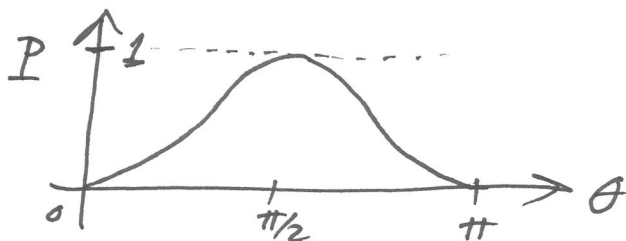
The eigenvectors of $\underline{\underline{\rho}}^{sc}$ are easy to find by inspection. Choose $\hat{\epsilon}_1^{(sc)}$ perpendicular to the plane containing $\hat{n}_0$ and $\hat{n}$ (the scattering plane) and then $\hat{\epsilon}_2^{(sc)}$ will be in that plane.
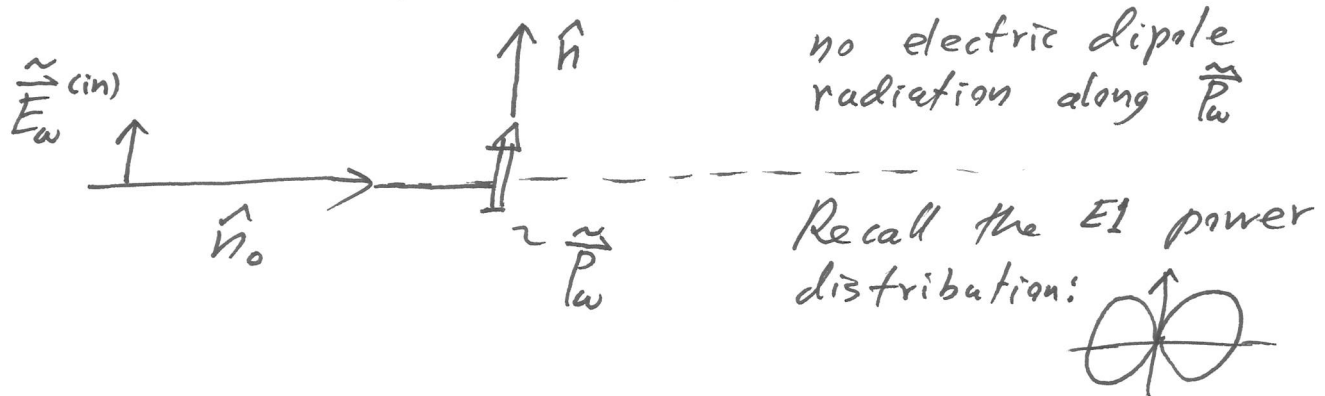

Then $\hat{n}_0 \cdot \hat{\epsilon}_1^{(sc)} = 0$ and $\hat{n}_0 \cdot \hat{\epsilon}_2^{(sc)} = -\sin\theta$

$$\underline{\underline{\rho}}^{sc} = \begin{pmatrix} \dfrac{1}{1+\cos^2\theta} & 0 \\ & \\ 0 & \dfrac{\cos^2\theta}{1+\cos^2\theta} \end{pmatrix}$$

The polarization is $P = \rho_1 - \rho_2 = \dfrac{1 - \cos^2\theta}{1+\cos^2\theta} = \dfrac{\sin^2\theta}{1+\cos^2\theta}$

Thus at a scattering angle of $\theta = \frac{\pi}{2}$, the scattered radiation is 100% polarized in the $\hat{\varepsilon}_1^{sc}$-direction (that is, perpendicular to the scattering plane). There is a simple explanation as to why there is a <u>maximum</u> in P and a <u>minimum</u> in $\sigma(\Omega)$ at $\frac{\pi}{2}$. For an incident wave polarized in the scattering plane, there can be <u>no</u> scattering at $\theta = \frac{\pi}{2}$:

no electric dipole radiation along $\tilde{\tilde{P}}_\omega$

Recall the E1 power distribution:

Since an unpolarized beam can be thought of as two equal intensity independent beams of orthogonal polarization, the beam above is <u>not</u> scattered. The other beam, polarized perpendicular to the page, will be scattered at $\theta = \frac{\pi}{2}$. Thus the cross section at $\theta = \frac{\pi}{2}$ will be a minimum while the polarization will be maximum.

Scattering from an Ensemble

If we have many scatterers rather than one and if the density is such that we can neglect multiple scattering, then the Green function for outgoing waves from the βth particle is

field point
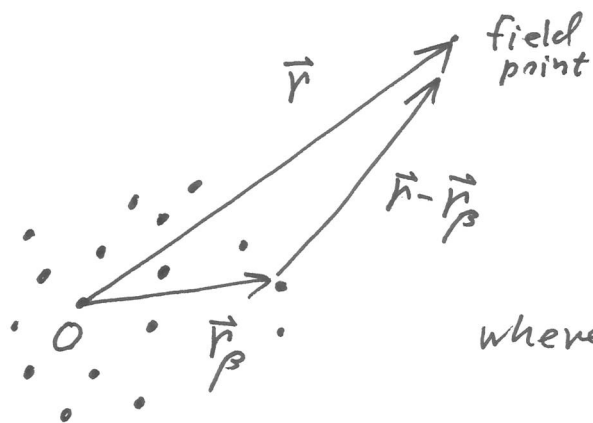
$$\tilde{G}_\omega(\vec{r}-\vec{r}_\beta-\vec{r}') = \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}_\beta-\vec{r}'|}}{|\vec{r}-\vec{r}_\beta-\vec{r}'|}$$

where $\vec{r}'$ ranges over the βth source.

Thus
$$\tilde{\vec{E}}_\omega^{(sc)}(\vec{r}) = \sum_\beta \iiint dV' e^{-i\vec{k}\cdot(\vec{r}'+\vec{r}_\beta)}\left[\tilde{\vec{J}}_\omega(\vec{r}')\right]_{tr} i\frac{\omega}{c}\frac{e^{i\frac{\omega}{c}r}}{r}$$

where $\vec{k} = \frac{\omega}{c}\hat{n}$, $\hat{n} = \frac{\vec{r}}{r}$, and $\vec{r}_\beta$ is the center of the βth molecule. We assume that all scatterers are identical so that there is no label β on the transverse current $\left[\tilde{\vec{J}}_\omega(\vec{r}')\right]_{tr}$. We also assume that the size of the collection of scatterers is small compared to the distance r from the center of the ensemble.

The incident wave at the βth scatterer is

$$\tilde{\vec{E}}_\omega^{(in)}(\vec{r}) = \tilde{\vec{E}}_\omega^{(in)} e^{i\vec{k}_0 \cdot \vec{r}_\beta}$$

and thus in the electric dipole approximation we have

$$\left[\tilde{\vec{E}}_\omega^{(sc)}(\vec{r})\right]_i = \frac{e^{i\frac{\omega}{c}r}}{r} \sum_{j=1}^{2} M_{ij} \tilde{E}_{\omega j}^{(in)} \sum_\beta e^{i(\vec{k}_0 - \vec{k}) \cdot \vec{r}_\beta}$$

where $M_{ij} = \frac{\mu_0}{4\pi} \omega^2 \alpha \, \hat{\epsilon}_i^{(sc)*} \cdot \hat{e}_j^{(in)}$ as before.

Thus the differential cross section is given by

$$\sigma(\Omega) = \sigma_0(\Omega) \, \mathcal{H}(\vec{q}) \qquad \vec{q} \equiv \vec{k}_0 - \vec{k}$$

where $\sigma_0(\Omega)$ is the differential cross section for a single scatterer and $\mathcal{H}(\vec{q}) = \sum_\beta \sum_r e^{i\vec{q} \cdot (\vec{r}_\beta - \vec{r}_r)}$

is a "structure factor" for the ensemble.

We next consider two extremes:

① a regular lattice (crystal)

② randomly arranged scatterers

① Scatterers on a lattice

In this case, $\vec{r}_\beta = \sum_{\ell=1}^{3} n_\ell \vec{a}_\ell$ where $n_1, n_2,$ and n_3 are integers (they specify the index β) and $\vec{a}_1, \vec{a}_2,$ and $\vec{a}_3$ are lattice vectors with the dimension of length. The scatterers in this case are a group of atoms or molecules called a unit cell. It is the unit cell which is repeated periodically throughout the lattice and can in itself be quite complex. The polarizability α would then pertain to the unit cell. While the lattice vectors $\vec{a}_\ell$ are linearly independent, they will not be orthogonal in general (only for a rectangular lattice). However, there exist reciprocal lattice vectors $\vec{b}_\ell$ such that

$$\vec{a}_\ell \cdot \vec{b}_m = \delta_{\ell m}. \qquad \text{Explicitly,}$$

$$\vec{b}_1 \equiv \frac{\vec{a}_2 \times \vec{a}_3}{\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3} \quad, \quad \vec{b}_2 \equiv \frac{\vec{a}_3 \times \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3} \quad, \quad \vec{b}_3 \equiv \frac{\vec{a}_1 \times \vec{a}_2}{\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3}$$

Suppose that we look at values of $\vec{q}$ such that

$$\vec{q} = 2\pi \sum_{m=1}^{3} \nu_m \vec{b}_m \qquad \text{where } \nu_m \text{ are integers}$$

then

$$\mathcal{I}(\vec{q}) = \sum_{\substack{n_1, n_2, n_3, \\ n_1', n_2', n_3'}} \exp\left[2\pi i \sum_{\ell=1}^{3} \sum_{m=1}^{3} \nu_m (n_\ell - n_\ell') \underbrace{\vec{b}_m \cdot \vec{a}_\ell}_{\delta_{\ell m}}\right]$$

$$= \sum_{\substack{n_1 n_2 n_3 \\ n_1' n_2' n_3'}} \exp\left[2\pi i \sum_{\ell=1}^{3} \nu_\ell (n_\ell - n_\ell')\right]$$

$$= \sum_{n_1 n_2 n_3} \sum_{n_1' n_2' n_3'} (1) = N^2$$

since each triple sum goes over the total number of scatterers N. So $\mathcal{I}(\vec{q}) = N^2$ when $\vec{q}$ is one of the possible wave vectors given on the bottom of the previous page and specified by the set of integers (positive or negative) ν_1, ν_2, ν_3.

For values of $\vec{q}$ different from this form, there is destructive interference and there is no radiation ($\mathcal{I}(\vec{q}) = 0$). Hence we only find diffracted radiation in certain directions, that is at certain __Bragg angles__. This is the basis of X-ray diffraction: (images).

the factor $\sigma_0(\Omega)$ serves as an additional structure factor which pertains to the internal structure of a unit cell. Thus we have

$$\sigma(\Omega) = \begin{cases} N^2 \sigma_0(\Omega) & \text{at Bragg angles} \\ 0 & \text{at other angles} \end{cases}$$

Note also that in the forward direction $(\vec{q} = 0)$ $\left(\text{also } \vec{k}_0 = \vec{k}\right)$ we again have $\sigma(0) = N^2 \sigma_0(0)$ while in the backward direction $\sigma(\pi) \approx 0$. This is very different from scattering from a single dipole where $\sigma_0(0) = \sigma_0(\pi)$.

---

② Randomly Arranged Scatterers

In this case $\mathcal{N}(\vec{q}) = \sum_\beta \sum_\gamma e^{i\vec{q}\cdot(\vec{r}_\beta - \vec{r}_\gamma)}$

$$= \sum_\beta (1) + \sum_{\beta \neq \gamma} \sum e^{i\vec{q}\cdot(\vec{r}_\beta - \vec{r}_\gamma)}$$

The factor $e^{i\vec{q}\cdot(\vec{r}_\beta - \vec{r}_\gamma)}$ for $\beta \neq \gamma$ takes on essentially all values on the unit circle in the

complex plane because of the large number of randomly located scatterers. Thus $\mathcal{T}_{(q)} = N$, and $\sigma(\Omega) = \sigma_0(\Omega) N$.

Note that in both cases ① and ②, the polarization is just that due to an individual scatterer, but in the Bragg case we must be looking at a Bragg angle.

Most sources emit more than one frequency of light and often a continuous distribution (spectrum) of frequencies (e.g. light bulbs and the Sun). The sources are incoherent at the different frequencies.

Scattering from a random ensemble of electric dipoles is called "Rayleigh scattering". The scattered power at a given frequency ω goes as ω^4 (and hence as $\frac{1}{\lambda^4}$). Thus shorter wavelengths are scattered out of the incident beam more efficiently. (Blue light is scattered more than red light.)

At sunrise and sunset, when sunlight (which is fairly flat in the visible spectrum and hence white) must travel through a lot of atmosphere, the sun appears red because blue light is scattered out of the beam. When you look at the sky in a direction away from the sun, the atmosphere is blue for the same reason. Furthermore, when you look at the zenith at sunrise and sunset, the sunlight should be strongly (ideally 100%) polarized.
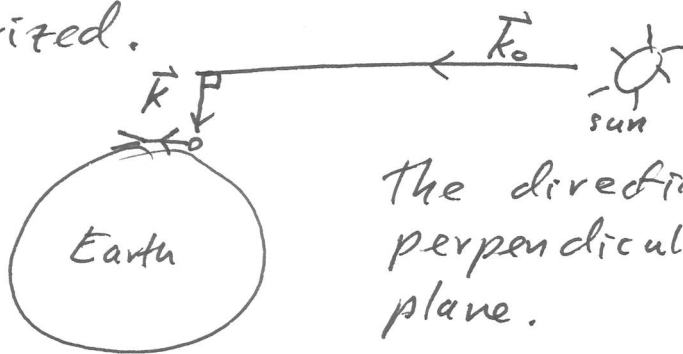
The direction of polarization is perpendicular to the scattering plane.

The actual polarization is less than 100% for a variety of reasons, one of which is multiple scattering. This is the same reason that clouds, milk, sugar, and snow look white. All wavelengths eventually get scattered so many times that there is no preference shown to blue light.