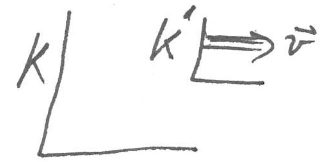
(video Terrell rotation)

③ Breakdown of Simultaneity



Look at two events which are simultaneous in K' but at different locations in that frame. Then $\Delta t' = 0$, $\Delta x' \neq 0$ and so

$$\Delta t = \gamma\left(0 + \frac{v\,\Delta x'}{V^2}\right) \neq 0$$

Thus two events which are simultaneous but spatially separated in one frame will <u>not</u> be simultaneous in another frame.

④ Causality and Limiting Velocity

Suppose that a "signal" is sent from x to $x+\Delta x$ in a time $\Delta t > 0$. Then the speed of the signal is $u = \dfrac{\Delta x}{\Delta t}$. Transforming to a frame K' moving with speed v relative to frame K we have

$$\Delta t' = \gamma\left(\Delta t - \frac{v\,\Delta x}{V^2}\right) = \gamma\Delta t\left(1 - \frac{vu}{V^2}\right).$$

We have already argued that $v < V$ (we assume $V^2 > 0$). Thus $\dfrac{v}{V} < 1$. But if $u > V$ then we can always find a frame such that

$$\frac{vu}{V^2} > 1 \quad \text{and hence} \quad \Delta t' < 0.$$

Thus the sending and receiving of events in K (which defines the time lapse Δt) will be reversed in K' — i.e. the receiving event will preceed the sending event and so causality will be violated. Since that is repugnant we reach the conclusion that no information can be sent with a speed greater than V.

(5) Velocity Addition along Direction of Motion

$$x' = \gamma(x - vt) \qquad\qquad dx' = \gamma(dx - v\,dt)$$

$$t' = \gamma\left(t - \frac{vx}{V^2}\right) \qquad\qquad dt' = \gamma\left(dt - \frac{v}{V^2}dx\right)$$

$$\Longrightarrow \frac{dx'}{dt'} \equiv u' = \frac{u - v}{1 - \frac{uv}{V^2}} \qquad \text{where } u \equiv \frac{dx}{dt}.$$

At this point, we use an experimental fact: The speed of light is the same in all inertial frames. Thus in the velocity transformation equation we have

$$u' = c \qquad\qquad u = c$$

$$c = \frac{c - v}{1 - \frac{cv}{V^2}} \quad \Longrightarrow \quad c - \frac{c^2 v}{V^2} = c - v$$

$$\Longrightarrow V^2 = c^2 \quad \Longrightarrow \quad \boxed{V = c}$$

Note that this is only one of many possible experiments that could be used to determine V.

⑥ General Lorentz Transformation

(i) Parallel Axes

Here we assume that the axes of K and K' are parallel as in the Lorentz transformation explored earlier, but the relative velocity between K and K' does not lie along the 1-axis. Since only the coordinate vector components along the velocity is affected we get

$$\vec{x}' = \gamma(\vec{x}_{\parallel} - \vec{v}t) + \vec{x}_{\perp}$$

$$t' = \gamma\left(t - \frac{\vec{x}\cdot\vec{v}}{c^2}\right)$$

where $\vec{x}_{\parallel} = \hat{n}(\hat{n}\cdot\vec{x})$ and $\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel} = \vec{x} - \hat{n}(\hat{n}\cdot\vec{x})$

and $\hat{n} \equiv \frac{\vec{v}}{v}$. Thus

$$\boxed{\begin{aligned} \vec{x}' &= (\gamma - 1)[\hat{n}(\hat{n}\cdot\vec{x}) - \vec{v}t] + \vec{x} - \vec{v}t \\ t' &= \gamma\left(t - \frac{\vec{x}\cdot\vec{v}}{c^2}\right) \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}}$$

(ii) Arbitrary Orientation of Axes

We simply rotate the right-hand side above in the $\vec{x}'$ equation by an arbitrary rotation matrix.

This can be done simply in dyadic notation. We go from a set of axes $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ to a set $\{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$ by

$$\overleftrightarrow{R} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_{ij}\, \hat{e}_i' \wedge \hat{e}_j$$

exterior product

Then we have

$$\boxed{\begin{aligned} \vec{x}' &= \overleftrightarrow{R} \cdot \left\{ (\gamma - 1)\left[\hat{n}(\hat{n}\cdot\vec{x}) - \vec{v}t\right] + \vec{x} - \vec{v}t \right\} \\ t' &= \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right) \end{aligned}}$$

7 General Time Dilation

When a moving clock accelerates, what happens to its proper clock rate? The usual assumption is that nothing happens to it. Thus a clock at rest in an inertial frame and a clock at rest in an accelerating frame "tick" in the same manner. This "clock hypothesis" can be experimentally verified by putting radioactive nuclei in a centrifuge and checking the observed life time against the inertial time dilation formula obtained in ①:

$$\tau = \frac{\tau_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Such experiments have been done and the clock hypothesis has been verified. So as a clock accelerates we can find its proper time lapse by transforming to an inertial frame which is instantaneously at rest with respect to the clock. Then

$$dt = \frac{dt_0}{\sqrt{1 - \frac{v^2(t)}{c^2}}} \quad \text{or} \quad \tau_0 = \int_{t=0}^{\tau} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$$

where τ_0 is the proper time read by the clock.

Suppose we have two clocks – one at the origin of K and the other at the origin of K'. These two frames coincide and are at rest at $t = 0 = t'$. Then K' accelerates in some arbitrary manner, goes out, and returns and comes to rest with respect to K at time τ (the two clocks again coinciding at that time). The clock in K' will read τ_0 while the clock in K will read τ.

The clock readings are related by

$$\tau_0 = \int_{t=0}^{\tau} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt < \tau$$

Moving clocks run slow.

<u>Homework</u>: Find the <u>general</u> velocity addition law for a general Lorentz transformation, with arbitrary orientation of axes.

4-tensor Notation

It is often convenient to introduce a notation which treats space and time in a similar fashion. First we introduce

$$x^0 \equiv ct \quad , \quad \beta \equiv \frac{v}{c} \quad \Longrightarrow \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Next we write the spatial components in a new notation $x^i \equiv \hat{e}_i \cdot \vec{x}$ (we use a superscript where before we had used a subscript. In the following, a subscript will take on new significance.)

The general Lorentz transformation obtained previously can be expressed in 4-vector notation as

$$x'^{\mu} = \ell^{\mu}{}_{\nu} x^{\nu} \qquad (\mu, \nu = 0, 1, 2, 3)$$

where we will use the Einstein summation convention of summing over repeated indices where one is up and the other is down.

<u>Homework</u> Determine the coefficients $\ell^{\mu}{}_{\nu}$ in terms of the rotation matrix coefficients R_{ij} and the velocity components.

What is it that characterizes the $\ell^{\mu}{}_{\nu}$?
Recall that for <u>rotations</u> we know that the <u>dot product</u> of two vectors is preserved under rotation — that is:

$$\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B} \qquad \text{where} \qquad \begin{aligned} \vec{A}' &= \overleftrightarrow{R} \cdot \vec{A} \\ \vec{B}' &= \overleftrightarrow{R} \cdot \vec{B} \end{aligned}$$

with $\overleftrightarrow{R}$ given previously. In matrix notation

$$\underline{A}'^{T} \underline{B}' = \underline{A}^{T} \underline{B} \qquad \text{where} \qquad \begin{aligned} \underline{A}' &= \underline{\underline{R}}\, \underline{A} \\ \underline{B}' &= \underline{\underline{R}}\, \underline{B} \end{aligned}$$

$$\Rightarrow \underline{\underline{R}}^{T} \underline{\underline{R}} = \underline{\underline{I}} \quad \text{or} \quad \underline{\underline{R}}^{T} = \underline{\underline{R}}^{-1} \quad (\underline{\underline{R}} \text{ is } \underline{\text{orthogonal}}).$$

What is the analog of this analysis for Lorentz transformations? Consider the special Lorentz transformation along the 1-axis for two events x^μ and y^μ.

$x'^1 = \gamma(x^1 - \beta x^0)$	$y'^1 = \gamma(y^1 - \beta y^0)$
$x'^2 = x^2$	$y'^2 = y^2$
$x'^3 = x^3$	$y'^3 = y^3$
$x'^0 = \gamma(x^0 - \beta x^1)$	$y'^0 = \gamma(y^0 - \beta y^1)$

We find that

$$x'^0 y'^0 - x'^1 y'^1 - x'^2 y'^2 - x'^3 y'^3 =$$

$$= \gamma^2 [x^0 y^0 - \beta(x^0 y^1 + x^1 y^0) + \beta^2 x^1 y^1]$$
$$\quad - \gamma^2 [x^1 y^1 - \beta(x^1 y^0 + x^0 y^1) - \beta^2 x^0 y^0] - x^2 y^2 - x^3 y^3$$
$$= \gamma^2 (1-\beta^2) [x^0 y^0 - x^1 y^1] - x^2 y^2 - x^3 y^3$$
$$= x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

So this combination is preserved under a Lorentz transformation, just like $a^1 b^1 + a^2 b^2 + a^3 b^3$ is preserved under a spatial rotation.

Introduce the "metric tensor"

$$g_{\mu\nu} \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\nu\mu}$$

There are actually two choices for the metric tensor. The one above is the "east coast" or "particle" metric, while $g'_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is the "west coast" or "gravity" metric. We will use the former.

With the definition of the metric tensor, the form which is invariant under a Lorentz transformation is

$$x'^{\mu} g_{\mu\nu} y'^{\nu} = x^{\mu} g_{\mu\nu} y^{\nu}$$

We define the lower index quantity

$$y_{\mu} \equiv g_{\mu\nu} y^{\nu}, \quad \text{that is} \quad y_0 = y^0 \,;\; y_i = -y^i$$

We use $g_{\mu\nu}$ to "lower an index." Then x^{μ} is called an contravariant component and

$x_{\mu} = g_{\mu\nu} x^{\nu}$ is called a covariant component.

In this notation, we have

$$x'^{\mu} y'_{\mu} = x^{\mu} y_{\mu} \qquad \text{(4-vector "dot" product)}$$

this is the analog of $\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$ under rotation.

Homework: Show explicitly that $x'^{\mu} y'_{\mu} = x^{\mu} y_{\mu}$ for a general Lorentz transformation with arbitrary rotation of axes.

The condition on the $\ell^{\mu}{}_{\nu}$ dictated by the analysis above is

$$(\ell^{\mu}{}_{\alpha} x^{\alpha}) g_{\mu\nu} (\ell^{\nu}{}_{\beta} y^{\beta}) = x^{\alpha} g_{\alpha\beta} y^{\beta}$$

and since x^{α} and y^{α} are arbitrary 4-vectors it must be that

$$\boxed{\ell^{\mu}{}_{\alpha} g_{\mu\nu} \ell^{\nu}{}_{\beta} = g_{\alpha\beta}}$$

It is also useful to introduce the "inverse of the metric tensor" $g^{\mu\nu}$ such that

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

4-dimensional Kronecker delta

Thus numerically

$$g^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = g^{\nu\mu}$$

which is the same as $g_{\mu\nu}$, but because of the summation convention we treat $g^{\mu\nu}$ and $g_{\mu\nu}$ as separate. For curved manifolds, upper and lower matter!

Now we can raise indices with $g^{\mu\nu}$ like this

$$x^{\mu} = g^{\mu\nu} x_{\nu}$$

For simplicity and efficiency, we can also define

$$l_{\mu\nu} = g_{\mu\alpha} l^{\alpha}{}_{\nu}$$

$$l_{\mu}{}^{\nu} = g_{\mu\alpha} g^{\nu\beta} l^{\alpha}{}_{\beta} \qquad (\text{e.g. } x'_{\mu} = l_{\mu}{}^{\nu} x_{\nu})$$

$$l^{\mu\nu} = g^{\nu\beta} l^{\mu}{}_{\beta}$$

The condition on a general Lorentz transformation now reads

$$\boxed{l_{\alpha\mu} l^{\alpha\nu} = \delta^{\nu}_{\mu}}$$

but keep in mind that $l^{\mu}{}_{\nu}$ are the coefficients defined originally.

<u>Homework</u>: Find explicitly the transformation coefficients $(l^{-1})^{\mu}{}_{\nu}$ for the inverse transformation

$$x^{\mu} = (l^{-1})^{\mu}{}_{\nu} x'^{\nu}$$

4-Tensors (constant)

A tensor is a generalization of a vector. Each index on a tensor transforms like a contravariant or covariant vector. Recall

$$x'^{\mu} = l^{\mu}{}_{\nu} x^{\nu}$$

$$x'_{\mu} = l_{\mu}{}^{\nu} x_{\nu}$$

Thus a mixed $(m+n)$-rank tensor (m contravariant and n covariant components) transforms as

$$T'^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} = l^{\mu_1}{}_{\alpha_1} \cdots l^{\mu_m}{}_{\alpha_m} l_{\nu_1}{}^{\beta_1} \cdots l_{\nu_n}{}^{\beta_n} T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n}$$

Tensors are very useful. For example, if we can show that a law of physics is of the form $C^{\mu} = 0$ (e.g. mechanics) or $A^{\mu\nu} = 0$, $B^{\mu\nu\lambda} = 0$ (e.g. E+M) where C^{μ}, $A^{\mu\nu}$, and $B^{\mu\nu\lambda}$ really are tensors under Lorentz transformations (that's a physical fact, not just a mathematical convenience), then in a different inertial frame

$$C'^{\mu} = 0 \quad , \quad A'^{\mu\nu} = 0 \quad , \quad B'^{\mu\nu\lambda} = 0$$

that is the physical law takes on the same form in all inertial frames — it is "form invariant" or "generally covariant" or "covariant".

That is a statement of Einstein's principle of relativity – the laws of physics take on the same form in all inertial frames.

Note that when we are dealing with tensor <u>fields</u> rather than constant tensors, we have

$$T'(x') = \mathcal{L}\, T(x)$$

where $x' = Lx \implies T'(x') = \mathcal{L}\, T(L^{-1}x')$

or replacing x' by x

$$\boxed{T'(x) = \mathcal{L}\, T(L^{-1}x)}$$

Here T stands for the tensor written out before and $\mathcal{L}$ stands for the product of the various $\ell^{\mu}{}_{\nu}$ matrices while L just means one factor of $\ell^{\mu}{}_{\nu}$. This expression gives the functional dependence of the tensor field on the variables (x^0, x^1, x^2, x^3).

Tensor indices can be raised and lowered just like vector indices

$$T^{\mu}{}_{\nu} = g_{\nu\alpha} T^{\mu\alpha} \qquad \text{and so forth}$$

Contractions

It is straightforward to show that if for example $T^{\mu\nu\lambda}$ is a tensor of rank 3, then $T^{\mu\nu}{}_{\nu}$ is a contravariant vector (rank 1). However unless $T^{\mu\nu\lambda}$ is totally symmetric, the vectors $T^{\mu\nu}{}_{\nu}$, $T^{\nu}{}_{\nu}{}^{\mu}$, and $T^{\nu\mu}{}_{\nu}$ are all different.

An important application is that of contraction to a Lorentz scalar. Consider the difference between two space time events

$$\Delta x^{\mu} = (x^{\mu})_a - (x^{\mu})_b$$

The quantity

$$\Delta s^2 \equiv \Delta x^{\mu} \Delta x_{\mu}$$

is clearly a scalar and hence is the same in all inertial frames. In particular

if $\Delta s^2 > 0$ then there is <u>no frame</u> in which the events are simultaneous. This is a time-like interval.

if $\Delta s^2 < 0$ Then we <u>can</u> find a frame in which the events are simultaneous. This is a space-like interval

if $\Delta s^2 = 0$ The two events can be connected by a light signal. This is a light-like interval.