B) Mechanics

The Newtonian law of motion in an inertial frame is

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = m\vec{u} \qquad \text{where } \vec{u} \text{ is the velocity of the particle}$$

$$\vec{u} = \frac{d\vec{r}}{dt}$$

The top equation is invariant under a Gallilean transformation, that is under $\vec{r}' = \vec{r} - \vec{v}t$.
But the principle of relativity is that the mathematical description of physical laws is unchanged under a Lorentz transformation. So we have to figure out how to generalize the <u>approximate</u> Law above (we know that $\vec{F} = \frac{d\vec{p}}{dt}$ works for speeds that are small compared to c). We should be able to express the Laws of mechanics in the language of 4-vectors. To begin with, how do we get a velocity 4-vector? The quantity $\frac{dx^\mu}{dt}$ is <u>not</u> a 4-vector since dx^μ is one but dt is not a scalar.
Consider an inertial frame in which a particle, whose dynamics we are trying to understand, is instantaneously

at rest. According to the clock hypothesis, the interval between two infinitesimally separated points along the spacetime trajectory (called the "worldline") of the particle is

$$ds^2 = c^2 d\tau^2$$

where $d\tau$ is the proper time lapse. But ds^2 is an ivariant so $ds^2 = c^2 d\tau^2$ in <u>any</u> inertial frame. Thus $d\tau$ is also an invariant. Recalling that

$$d\tau = \frac{dt}{\gamma} = dt\sqrt{1-\frac{u^2}{c^2}}$$

we can construct a 4-vector velocity (or "4-velocity")

$$\mathbb{U}^\mu = \frac{dx^\mu}{d\tau} = \begin{cases} \gamma c \;, & \text{for } \mu = 0 \\ \gamma u^i \;, & \text{for } \mu = 1,2,3 \end{cases}$$

$$u^i = \frac{dx^i}{dt}$$

Note that $\mathbb{U}^\mu \mathbb{U}_\mu = c^2$. We define the 4-momentum of the particle as

$$\mathbb{P}^\mu = m\,\mathbb{U}^\mu = \begin{cases} \mathbb{P}^0 = m\gamma c \\ \vec{\mathbb{P}} = m\gamma\vec{u} \end{cases}$$

For $u \ll c$, we have $\gamma \simeq 1$ and $\vec{\mathbb{P}} \simeq m\vec{u}$ which is the non-relativistic momentum. To see the significance of $\mathbb{P}^0$, we keep one more term in the small velocity expansion of γ:

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \simeq 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

$$\Rightarrow P^0 = mc + mc\,\frac{1}{2}\frac{u^2}{c^2} + \cdots$$

$$= \frac{1}{c}\left[mc^2 + \frac{1}{2}mu^2 + \cdots\right]$$

Thus $P^0 c$ is the kinectic energy to within an additive constant mc^2 – the so-called "rest energy."

This interpretation of P^0 and $\vec{P}$ can be seen further in the generalization of the work-energy theorem. Since P^μ is a 4-vector, then $\frac{dP^\mu}{d\tau}$ is also a 4-vector. Hence we generalize

$\vec{F} = \frac{d\vec{p}}{dt}$ by $\quad F^\mu = \frac{dP^\mu}{d\tau} \quad$ where

F^μ is the 4-vector force or 4-force. Now

$$U_\mu F^\mu = U_\mu \frac{dP^\mu}{d\tau} = m U_\mu \frac{dU^\mu}{d\tau}$$

$$= \frac{m}{2}\frac{d}{d\tau}\left(U_\mu U^\mu\right) = \frac{m}{2}\frac{d}{d\tau}(c^2) = 0$$

Thus

$$U_\mu \frac{d\mathbb{P}^\mu}{d\tau} = 0 = \gamma^2 \left[\frac{d}{dt}(c\mathbb{P}^0) - \vec{u}\cdot\frac{d}{dt}\vec{\mathbb{P}} \right]$$

spatial part of $\mathbb{P}^\mu$ 4-momentum

$$\Rightarrow \quad \frac{d}{dt}(c\mathbb{P}^0) = \vec{u}\cdot\vec{\mathfrak{F}}$$

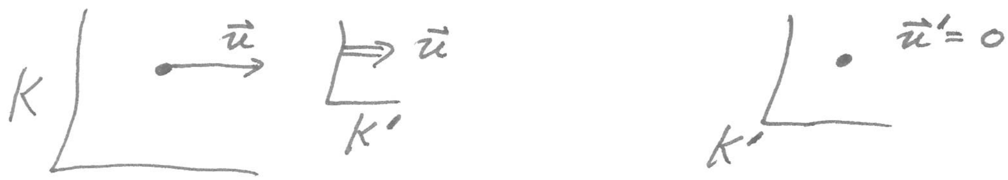
where we have written $\vec{\mathfrak{F}} = \frac{d\vec{\mathbb{P}}}{dt}$. Note that $\vec{\mathfrak{F}}$ is not the spatial part of the 4-force $\mathbb{F}^\mu$ (but $\gamma\vec{\mathfrak{F}}$ is). $\vec{\mathfrak{F}}$ is, however, the analog of the Newtonian force $\vec{F}$. Thus we may interpret $\vec{u}\cdot\vec{\mathfrak{F}}$ as the rate at which work is done in a particular inertial frame. Then $\frac{d}{dt}(c\mathbb{P}^0)$ is the rate of change of kinetic energy in that same frame. Again mc^2 is just an additive constant which is independent of time and does not affect the work-energy theorem.

Since $\mathbb{P}^\mu$ is a 4-vector, then $\mathbb{P}^\mu\mathbb{P}_\mu = m^2c^2$ is an invariant and thus

$$(\mathbb{P}^0)^2 = |\vec{\mathbb{P}}|^2 + m^2c^2 \qquad \text{and}$$

$$E \equiv c\mathbb{P}^0 = \sqrt{|\vec{\mathbb{P}}|^2c^2 + m^2c^4} \qquad \text{is the energy}$$

Also, if we make a Lorentz transformation to the frame in which the particle is instantaneously at rest



then the momentum transforms as

$$\vec{\mathbb{P}}' = 0 = \gamma\left(\vec{\mathbb{P}} - \frac{\vec{u}}{c}\mathbb{P}^0\right)$$

$$\Rightarrow \vec{u} = \frac{\vec{\mathbb{P}}c}{\mathbb{P}^0} = \frac{\vec{\mathbb{P}}c^2}{E}$$

We also note that

$$\vec{\nabla}_{\vec{\mathbb{P}}} E = \vec{\nabla}_{\vec{\mathbb{P}}}\left(\sqrt{|\vec{\mathbb{P}}|^2 c^2 + m^2c^4}\right) = \frac{\vec{\mathbb{P}}c^2}{E} = \vec{u}$$

and this result holds both non-relativistically and relativistically. Also note that for $m=0$, we have

$$E = |\vec{\mathbb{P}}|c \qquad \text{and thus} \qquad u = c.$$

massless particles move at the speed of light.

4-Momentum Conservation Law

Consider a collision between particles 1 and 2. In Newtonian Mechanics we have in an <u>elastic</u> collision (where b = before and a = after)

$$\vec{P}_{a1} + \vec{P}_{a2} = \vec{P}_{b1} + \vec{P}_{b2} \qquad \vec{p} = m\vec{u} \qquad \text{Linear Momentum}$$

$$\tfrac{1}{2} m_1 u_{a1}^2 + \tfrac{1}{2} m_2 u_{a2}^2 = \tfrac{1}{2} m_1 u_{b1}^2 + \tfrac{1}{2} m_2 u_{b2}^2 \qquad \text{kinetic energy}$$

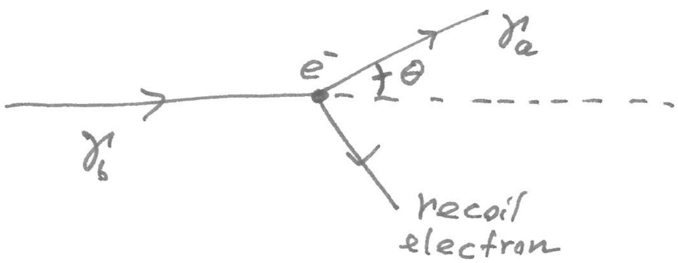
$$m_{1a} = m_{1b} \qquad m_{2a} = m_{2b} \qquad \text{mass}$$

The forces during the collision are assumed to be short ranged. These conservation laws can be written as the low velocity limit of the relativistic

$$\mathbb{P}_{a1}^{\mu} + \mathbb{P}_{a2}^{\mu} = \mathbb{P}_{b1}^{\mu} + \mathbb{P}_{b2}^{\mu} \qquad (\mu = 0, 1, 2, 3)$$

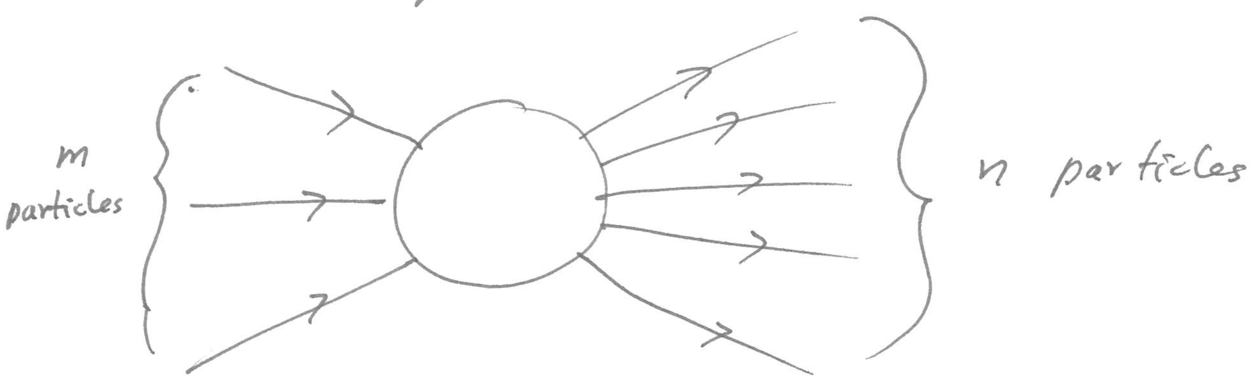
This law must then be true in <u>any</u> inertial frame since it is a statement among 4-vectors and is preserved under a Lorentz transformation.

<u>Homework!</u> Consider Compton scattering in which a photon of energy E_γ is scattered off of an electron initially at rest. The scattering angle is θ. Find the energy of the outgoing photon as a function of scattering angle.



Repeat the calculation for the case in which the electron has an initial momentum of $100\, m_e c$ directed opposite the photon momentum. This "inverse Compton scattering" is important in astrophysics.

We now hypothesize that in any process in which there are particles which are free before and after a short-range interaction then the <u>total</u> 4-momentum is conserved. (The initial and final particles are different in general.)



$$\mathbb{P}_{b1}^{\mu} + \dots + \mathbb{P}_{bm}^{\mu} = \mathbb{P}_{a1}^{\mu} + \dots + \mathbb{P}_{an}^{\mu} \qquad \left(\begin{array}{c} m \neq n \\ \text{necessarily} \end{array}\right)$$

In a non-relativistic inelastic collision, the spatial part of the relation above is valid, but energy is clearly not conserved, but in Newtonian mechanics that energy is kinetic energy.

the relativistic conservation law above for $\mu = 0$ involves rest energy as well as kinetic energy so that combination can be conserved, in fact is conserved as experiment shows. Thus rest energy does have physical significance in the context of inelastic collisions.

As an example, we consider the decay of a K° meson into π^+ and π^- mesons. We look at the decay of a K° at rest.

space part of 4-momentum

$$0 = \vec{P}_+ + \vec{P}_- \qquad \text{Define: } \vec{P} = \vec{P}_+ = -\vec{P}_-$$

$$E_{K^\circ} = M_{K^\circ} c^2 = E_+ + E_- = 2\sqrt{P^2 c^2 + m_\pi^2 c^4}$$

$$\Rightarrow \frac{m_{K^\circ}^2 c^2}{4} = P^2 + m_\pi^2 c^2$$

$$\Rightarrow P = \frac{c}{2}\sqrt{m_{K^\circ}^2 - 4 m_\pi^2}$$

and hence the velocity of the π mesons is

$$v_\pi = \frac{Pc^2}{E_\pi} = \frac{Pc^2}{\frac{1}{2} M_{K^\circ} c^2} = \frac{2P}{m_{K^\circ}}$$

$$\Rightarrow \beta_\pi = \frac{v_\pi}{c} = \sqrt{1 - \left(\frac{2 m_\pi}{m_{K^\circ}}\right)^2}$$

Putting in $m_\pi c^2 = 140\ MeV$ and $m_{K^0} c^2 = 498\ MeV$

we find

$$\beta_\pi = 0.83\,c$$

As a second example, we consider the problem of thresholds for particular reactions which are not energetically possible below a certain total energy.

Consider the reaction

$$p^+ + p^+ \longrightarrow d^+ + \pi^+$$

Assume that one of the protons is at rest in the lab (for example, in a liquid hydrogen target) while the other proton is given a certain kinetic energy K_L in the laboratory by an accelerator. Define the 4-momenta:

P^μ — projectile proton

T^μ — target proton

D^μ — deuteron

π^μ — pion

So 4-momentum conservation is

$$P^\mu + T^\mu = D^\mu + \pi^\mu$$

Contract both sides to scalars

$$(P^\mu + T^\mu)(P_\mu + T_\mu) = (D^\mu + \pi^\mu)(D_\mu + \pi_\mu)$$

Each side of the equation is a scalar and hence Lorentz invariant, so they can be evaluated in <u>any</u> inertial frame. In fact, each side can be evaluated in a <u>different</u> inertial frame.

We evaluate the left-hand side in the lab frame where

$$T^\mu = (m_p c, 0, 0, 0)$$

so

$$\begin{aligned}(P^\mu + T^\mu)(P_\mu + T_\mu) &= P^\mu P_\mu + 2P^\mu T_\mu + T^\mu T_\mu \\ &= m_p^2 c^2 + 2P^0 m_p c + m_p^2 c^2 \\ &= 2m_p^2 c^2 + 2\left(m_p c + \frac{K_L}{c}\right) m_p c \\ &= 4 m_p^2 c^2 + 2 K_L m_p\end{aligned}$$

We evaluate the right-hand side in the center of momentum frame in which $\vec{D} + \vec{\pi} = 0$ and since we are interested in the <u>minimum</u> value of K_L to produce a deuteron and a pion then we further want $\vec{D} = 0 = \vec{\pi}$. That is, we want the d^+ and π^+ created <u>at rest</u> in the c-o-m frame (they will be moving in the $\vec{P}$ direction in the lab frame.)

$$(D^\mu + \pi^\mu)(D_\mu + \pi_\mu) = (m_d + m_\pi)^2 c^2$$

$$\Rightarrow 4 m_p^2 c^2 + 2 (K_L)_{min} m_p = (m_d + m_\pi)^2 c^2$$

$$\Rightarrow (K_L)_{min} = \frac{(m_d + m_\pi)^2 c^2 - 4 m_p^2 c^2}{2 m_p} = 2 m_p c^2 \left[\left(\frac{m_d + m_\pi}{2 m_p} \right)^2 - 1 \right]$$

For $m_p c^2 = 938.27$ MeV, $m_d c^2 = 1875.61$ MeV
$m_\pi c^2 = 139.57$ MeV we find

$(K_L)_{min} = 287.5$ MeV So when the laboratory beam of protons reaches this kinetic energy, we can produce a π^+ along with a deuteron. At slightly higher energy, we can produce a π^+ meson with unbound nucleons ($p^+ + n^0$).

Homework! If m_p = mass of a projectile particle and m_t = mass of a target particle, show that the minimum kinetic energy of the projectile in the lab frame (with the target at rest) required to produce a particular set of reaction products is

$$(K_L)_{min} = \frac{M_{out}^2 c^2 - M_{in}^2 c^2}{2M_t}$$

where M_{out} = sum of the masses of the outgoing particles and $M_{in} = M_p + M_t$

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c) The Electromagnetic Field

Tensors through differentiation

Suppose $\Psi_{(x)}$ is a scalar field under Lorentz transformations. We then construct a 4-component field $V_\mu(x) = \partial_\mu \Psi_{(x)}$ where $\partial_\mu = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$.

Is this quantity $V_\mu(x)$ a vector field? To check this, we write in the $\bar{K}$ frame:

$$\bar{V}_\mu(\bar{x}) = \bar{\partial}_\mu \bar{\Psi}(\bar{x}) \underset{\uparrow}{=} \bar{\partial}_\mu \Psi_{(x)} = \partial_\beta \Psi_{(x)} \frac{\partial x^\beta}{\partial \bar{x}^\mu} = \frac{\partial x^\beta}{\partial \bar{x}^\mu} V_\beta(x)$$

where we used the fact that $\bar{\Psi}(\bar{x}) = \Psi_{(x)}$ since $\Psi_{(x)}$ is a scalar field.

Now

$$\bar{x}^\alpha = \ell^\alpha{}_\mu x^\mu \quad \Longrightarrow \quad \frac{\partial \bar{x}^\alpha}{\partial x^\mu} = \ell^\alpha{}_\mu$$

and

$$x^\beta = (\ell^{-1})^\beta{}_\mu \bar{x}^\mu \quad \Longrightarrow \quad \frac{\partial x^\beta}{\partial \bar{x}^\mu} = (\ell^{-1})^\beta{}_\mu$$

The defining property for a Lorentz transformation is

$$\ell^\mu{}_\nu \, \ell_\mu{}^\beta = \delta^\beta_\nu$$

but also by definition $(\ell^{-1})^\beta{}_\mu \ell^\mu{}_\nu = \delta^\beta_\nu$

$$\Longrightarrow (\ell^{-1})^\beta{}_\mu = \ell_\mu{}^\beta$$

So $\dfrac{\partial x^\beta}{\partial \bar{x}^\mu} = \ell_\mu{}^\beta \quad \Longrightarrow \quad \bar{V}_\mu(\bar{x}) = \ell_\mu{}^\beta V_\beta(x)$

That is, $V_\mu(x)$ transforms like a covariant vector.

If we define $\partial^\mu = g^{\mu\nu}\partial_\nu = \left(\dfrac{\partial}{\partial x^0}, -\vec{\nabla}\right)$ then

$\partial^\mu \psi(x) = V^\mu(x)$ is a contravariant vector field.

In a similar manner, differentiation of a tensor field by ∂_μ or ∂^μ creates a tensor field with rank higher by 1 covariant or contravariant index, respectively.

Maxwell Equations in Vacuum

(We are going to use c.g.s. units to match Jackson.)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Recall that the fields can be written in terms of the potentials Φ and $\vec{A}$ as

$$E^i = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial}{\partial x^i}\Phi = -\partial_0 A^i - \partial_i \Phi = -\partial^0 A^i + \partial^i \Phi$$

Also

$$B^i = \sum_{jk} \epsilon^{ijk}\partial_j A^k = -\sum_{jk}\epsilon^{ijk}\partial^j A^k$$

$$B^1 = -\partial^2 A^3 + \partial^3 A^2$$

$$B^2 = -\partial^3 A^1 + \partial^1 A^3$$

$$B^3 = -\partial^1 A^2 + \partial^2 A^1$$

Suppose we define $A^0 \equiv \Phi$ and then we define

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -F^{\nu\mu} \qquad \text{antisymmetric}$$

As a matrix

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

Since $F^{\mu\nu}$ is a 4×4 antisymmetric matrix, it has 6 independent components.

Next we define $J^0 = c\rho$. Then the Mawell Equations are written:

$$\boxed{\begin{aligned} &\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \\ &\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0 \end{aligned}}$$

Homework! Show that these equations give Maxwell's equations.

It is tempting to say that all the quantities defined above — A^μ, $F^{\mu\nu}$, J^μ — are tensor fields of the appropriate rank, but that is a matter of physics not mathematics or definition. Next we look at the quantities in question and show that they really are 4-tensors.