Waveguides and Resonant Cavities

Consider a permeable dielectric cylinder (ϵ, μ real $\Rightarrow$ no absorption) of arbitrary cross section surrounded by a perfect conductor (conductivity $\tilde{\sigma} = \infty$).

rectangular wave guide — cylindrical waveguide — coaxial cable — transmission line

waveguides $\Rightarrow$ infinite length

resonant cavities $\Rightarrow$ finite length

There will be no fields in the conductor since $\vec{J} = \tilde{\sigma}\vec{E}$

$$\vec{E} = \frac{J}{\tilde{\sigma}} = 0 \quad \text{if} \quad \tilde{\sigma} = \infty.$$

Source-free Maxwell Equations $\quad \rho = 0$, $\vec{J} = 0$

$$\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \qquad \vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0$$

$$\vec{\nabla} \times \vec{B}(\vec{r},t) = \frac{\mu\epsilon}{c}\frac{\partial \vec{E}(\vec{r},t)}{\partial t} \qquad \vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0$$

(c.g.s. units)

Look for normal mode solutions of the form

$$\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})\, e^{-i\omega t}\right]$$

$$\vec{\mathcal{B}}(\vec{r},t) = \mathrm{Re}\left[\vec{B}(\vec{r})\, e^{-i\omega t}\right]$$

Then

$$\vec{\nabla}\times\vec{E}(\vec{r}) = \frac{i\omega}{c}\vec{B}(\vec{r}) \qquad \vec{\nabla}\cdot\vec{E}(\vec{r}) = 0$$

$$\vec{\nabla}\times\vec{B}(\vec{r}) = -\frac{i\omega}{c}\mu\epsilon\,\vec{E}(\vec{r}) \qquad \vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$$

From which we obtain the vector Helmholtz equations:

$$\left(\nabla^2 + \frac{\omega^2}{v^2}\right)\vec{E}(\vec{r}) = 0 \qquad \left(\nabla^2 + \frac{\omega^2}{v^2}\right)\vec{B}(\vec{r}) = 0$$

with $v = \dfrac{c}{\sqrt{\epsilon\mu}}$ (c.g.s. units)

The Boundary Conditions are

surface charge density

$$\hat{n}\times\vec{E}(\vec{r})\Big|_S = 0 \qquad \hat{n}\cdot\vec{E}(\vec{r})\Big|_S = \frac{4\pi}{\epsilon}\sigma(\vec{r})\Big|_S$$

$$\hat{n}\cdot\vec{B}(\vec{r})\Big|_S = 0 \qquad \hat{n}\times\vec{B}(\vec{r})\Big|_S = \frac{4\pi\mu}{c}\vec{K}(\vec{r})\Big|_S$$

$\hat{n}$ perpendicular to surface (no z component)

surface current density

Choose $\hat{e}_z$ along the waveguide axis.

$$\vec{\nabla} = \vec{\nabla}_{tr} + \hat{e}_z\frac{\partial}{\partial z}$$

where "tr" means transverse (think of tr as x and y)

$$\vec{E}(\vec{r}) = \vec{E}_{tr}(\vec{r}) + \hat{e}_z E_z(\vec{r})$$

$$\vec{B}(\vec{r}) = \vec{B}_{tr}(\vec{r}) + \hat{e}_z B_z(\vec{r})$$

$$E_z(\vec{r}) = \hat{e}_z\cdot\vec{E}(\vec{r})$$

$$\vec{E}_{tr}(\vec{r}) = \vec{E}(\vec{r}) - \hat{e}_z E_z(\vec{r}) = \left[\hat{e}_z\times\vec{E}(\vec{r})\right]\times\hat{e}_z$$

Separate Maxwell's equations into transverse and longitudinal terms

$$\boxed{\vec{\nabla}_{tr} \cdot \vec{E}_{tr}(\vec{r}) = -\frac{\partial}{\partial z} E_z(\vec{r})} \qquad \boxed{\vec{\nabla}_{tr} \cdot \vec{B}_{tr}(\vec{r}) = -\frac{\partial}{\partial z} B_z(\vec{r})}$$

$$\vec{\nabla} \times \vec{E} = \frac{i\omega}{c} \vec{B}$$

$$\Rightarrow \left(\vec{\nabla}_{tr} + \hat{e}_z \frac{\partial}{\partial z}\right) \times \left(\vec{E}_{tr} + \hat{e}_z E_z\right) = \frac{i\omega}{c}\left(\vec{B}_{tr} + \hat{e}_z B_z\right) \qquad (*)$$

$$\hat{e}_z \cdot (*) \Rightarrow \boxed{\hat{e}_z \cdot \left(\vec{\nabla}_{tr} \times \vec{E}_{tr}\right) = \frac{i\omega}{c} B_z}$$

$$\hat{e}_z \times (*) \Rightarrow \boxed{\vec{\nabla}_{tr} E_z - \frac{\partial}{\partial z} \vec{E}_{tr} = \frac{i\omega}{c} \hat{e}_z \times \vec{B}_{tr}}$$

Similarly $\vec{\nabla} \times \vec{B} = -\frac{i\omega}{c} \mu\epsilon \vec{E}$

$$\Rightarrow \boxed{\hat{e}_z \cdot \left(\vec{\nabla}_{tr} \times \vec{B}_{tr}\right) = -\frac{i\omega}{c} \mu\epsilon E_z}$$

$$\Rightarrow \boxed{\vec{\nabla}_{tr} B_z - \frac{\partial}{\partial z} \vec{B}_{tr} = -\frac{i\omega}{c} \mu\epsilon\, \hat{e}_z \times \vec{E}_{tr}} \qquad (**)$$

B.C. $\hat{n} \times \vec{E}\big|_s = 0 \Longrightarrow \boxed{E_z\big|_s = 0}$ and $\hat{n} \times \vec{E}_{tr}\big|_s = 0$

Proof:
$$\hat{n} \times \vec{E} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ n_x & n_y & 0 \\ E_x & E_y & E_z \end{vmatrix} = \begin{array}{l} \hat{e}_x n_y E_z \\ + \hat{e}_y n_x E_z \\ + \hat{e}_z (n_x E_y - n_y E_x) \end{array} = \begin{array}{r} 0 \\ +0 \\ +0 \end{array}$$

B.C. $\hat{n}\cdot\vec{B}\Big|_s = 0 \implies \hat{n}\cdot\vec{B}_{tr}\Big|_s = 0$ $\left(\begin{matrix}\hat{n}\text{ already}\\ \perp\text{ to }\hat{e}_z\end{matrix}\right)$

$$\hat{n}\cdot(**)\Big|_s \implies \hat{n}\cdot\left(\vec{\nabla}_{tr} B_z\right)\Big|_s - \underbrace{\hat{n}\cdot\frac{\partial\vec{B}_{tr}}{\partial z}\Big|_s}_{\substack{0\text{ since}\\ =\frac{\partial}{\partial z}(\hat{n}\cdot\vec{B}_{tr})\text{ but}\\ \hat{n}\cdot\vec{B}_{tr}=0\text{ from}\\ \text{above}}} = \underbrace{\frac{-i\omega}{c}\mu\epsilon\,\hat{n}\cdot(\hat{e}_z\times\vec{E}_{tr})\Big|_s}_{\substack{0\text{ since}\\ =\hat{e}_z\cdot(\vec{E}_{tr}\times\hat{n})|_s\\ \text{but }\vec{E}_{tr}\times\hat{n}=0\\ \text{from previous page}\\ \vec{E}_{tr}\parallel\hat{n}\text{ on boundary}}}$$

$$\implies \hat{n}\cdot\left(\vec{\nabla}_{tr} B_z\right)\Big|_s = 0$$

and we can write this as $\boxed{\hat{n}\cdot\left(\vec{\nabla} B_z\right)\Big|_s = 0}$

since $\hat{n}\perp\hat{e}_z$.

These two boundary conditions for the z-components of the fields look like

Dirichlet: $E_z(\vec{r})\Big|_s = 0$

and Neumann: $\hat{n}\cdot\left[\vec{\nabla} B_z(\vec{r})\right]\Big|_s = 0$

conditions that we saw last semester

TEM waves (transverse electromagnetic)

Suppose $E_z(\vec{r}) = 0 = B_z(\vec{r})$ everywhere, not just on the boundary

Then the boundary conditions are automatically satisfied.

Also:

$$\vec{\nabla} \cdot \vec{E}_{tr}(\vec{r}) = 0 \qquad \vec{\nabla} \cdot \vec{B}_{tr}(\vec{r}) = 0$$

$$\vec{\nabla}_{tr} \times \vec{E}_{tr}(\vec{r}) = 0 \qquad \vec{\nabla}_{tr} \times \vec{B}_{tr}(\vec{r}) = 0$$

(see page 3 for these last two equations)

$$-\frac{\partial}{\partial z} \vec{E}_{tr}(\vec{r}) = \frac{i\omega}{c} \hat{e}_z \times \vec{B}_{tr}(\vec{r})$$

$$-\frac{\partial}{\partial z} \vec{B}_{tr}(\vec{r}) = -\frac{i\omega}{c} \mu\epsilon \, \hat{e}_z \times \vec{E}_{tr}(\vec{r})$$

We will look for solutions of the form

$$\vec{E}_{tr}(\vec{r}) = \vec{E}_{tr}(\vec{s})\, e^{ikz}$$

$$\vec{B}_{tr}(\vec{r}) = \vec{B}_{tr}(\vec{s})\, e^{ikz}$$

where $\vec{s}$ is the radial cylindrical polar coordinate $\{s, z, \varphi\}$

$$\Rightarrow -k \vec{E}_{tr}(\vec{s}) = \frac{\omega}{c} \hat{e}_z \times \vec{B}_{tr}(\vec{s})$$

$$k \vec{B}_{tr}(\vec{s}) = \frac{\omega}{c} \mu\epsilon \, \hat{e}_z \times \vec{E}_{tr}(\vec{s})$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \mu\epsilon \qquad k = \frac{\omega}{v} \qquad v = \frac{c}{\sqrt{\mu\epsilon}}$$

The wave vector k is real for all $\omega \Rightarrow$ there is no "cutoff" frequency.

If we define $\Phi(\vec{s})$ such that

$$\vec{E}_{tr}(\vec{s}) = -\vec{\nabla}_{tr}\,\Phi(\vec{s})$$

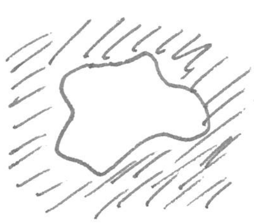
then $\vec{\nabla}_{tr} \times \vec{E}_{tr}(\vec{s}) = 0$ is automatically satisfied

and $\vec{\nabla} \cdot \vec{E}_{tr} = 0 \Longrightarrow \nabla^2_{tr}\,\Phi(\vec{s}) = 0$

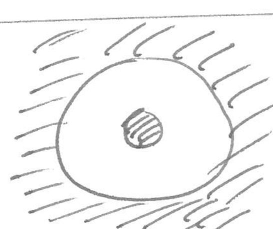
and this is a two dimensional Laplace's equation for electro<u>statics</u>.

Notice that a right cylinder will not support a TEM wave since $\Phi(\vec{s})\big|_s =$ constant on the boundary, then

$\Phi(\vec{s}) =$ constant everywhere within $\Longrightarrow \vec{E}_{tr}(\vec{s}) = 0$



$\leftarrow \Phi =$ constant on the conductor surface

But a geometry like  can support a TEM wave by placing a potential difference between the <u>two</u> conductors. Then find

$$\vec{E}_{tr}(\vec{s}) = -\vec{\nabla}_{tr}\,\Phi(\vec{s}) \quad \text{and} \quad \vec{B}_{tr}(\vec{s}) = \sqrt{\mu\epsilon}\,\hat{\epsilon}_z \times \vec{E}_{tr}(\vec{s})$$

TM waves transverse magnetic

Here $B_z(\vec{r}) = 0$ everywhere in the volume of the wave guide. Then $\hat{n}\cdot\left[\vec{\nabla} B_z(\vec{r})\right]\Big|_s = 0$ is automatically satisfied on the surface (Neumann B.C.)

And of course $E_z(\vec{r}) = 0\Big|_s$ on the surface.

We will look for solutions of the form:

$$\vec{B}_{tr}(\vec{r}) = \vec{B}_{tr}(\vec{s})\, e^{ikz}$$

$$\vec{E}_{tr}(\vec{r}) = \vec{E}_{tr}(\vec{s})\, e^{ikz}$$

$$E_z(\vec{r}) = \psi(\vec{s})\, e^{ikz}$$

The Helmholtz equation for $E_z(\vec{r})$ is

$$\left(\nabla^2 + \frac{\omega^2}{v^2}\right) E_z(\vec{r}) = 0$$

$$\Rightarrow \left(\nabla_{tr}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v^2}\right)\left[\psi(\vec{s})\, e^{ikz}\right] = 0$$

$$\Rightarrow \left(\nabla_{tr}^2 - k^2 + \frac{\omega^2}{v^2}\right)\psi(\vec{s}) = 0$$

with Dirichlet B.C. $\psi(\vec{s})\Big|_s = 0$

Define $\gamma^2 = \frac{\omega^2}{v^2} - k^2 = \frac{\omega^2}{c^2}\mu\epsilon - k^2$

$$(\nabla^2_{tr} + \gamma^2)\,\psi(\vec{s}) = 0$$

$\psi(\vec{s})$ must be oscillatory (in x and y - Cartesian) (or in s and φ - Cylindrical) since $\gamma^2 > 0$

This is a 2-dimensional eigenvalue problem

$$\Longrightarrow \gamma = \gamma_{mn} \quad \text{for the } m,n^{th} \text{ mode.}$$

Suppose that we have already solved for $\psi(\vec{s})$ and thus we know $E_z(\vec{r})$. Then

$$\vec{\nabla}_{tr} \times \vec{E}_{tr}(\vec{s}) = 0 \quad \Longrightarrow \quad \vec{E}_{tr}(\vec{s}) = -\vec{\nabla}_{tr}\,\Phi(\vec{s})$$

$$\vec{\nabla}_{tr} \cdot \vec{E}_{tr}(\vec{s}) = -\frac{\partial E_z}{\partial z} \quad \Longrightarrow \quad \nabla^2_{tr}\,\Phi(\vec{s}) = ik\,\psi(\vec{s})$$

Try a solution $\Phi(\vec{s}) = A\,\psi(\vec{s})$ and find A.

$$\nabla^2_{tr}\,\psi(\vec{s}) = -\gamma^2\psi(\vec{s}) = \frac{1}{A}\,ik\,\psi(\vec{s}) \quad \Longrightarrow \quad A = \frac{-ik}{\gamma^2}$$

So $\vec{E}_{tr}(\vec{s}) = \frac{ik}{\gamma^2_{mn}}\,\vec{\nabla}_{tr}\,\psi(\vec{s})$

(**) $$\cancelto{0}{\vec{\nabla}_{tr} B_z(\vec{r})} - \frac{\partial}{\partial z} \vec{B}_{tr}(\vec{r}) = -\frac{i\omega}{c}\mu\epsilon \; \hat{e}_z \times \vec{E}_{tr}(\vec{r})$$

$$\Rightarrow \vec{B}_{tr}(\vec{s}) = \frac{\omega}{c}\frac{\mu\epsilon}{k}\, \hat{e}_z \times \vec{E}_{tr}(\vec{s})$$

$$\equiv \frac{\mu}{Z_{TM}}\, \hat{e}_z \times \vec{E}_{tr}(\vec{s})$$

where $Z_{TM} = \frac{kc}{\epsilon\omega}$ TM wave impedance

$$= \frac{\mu}{Z_{TM}}\, \hat{e}_z \times \left[\frac{ik}{\gamma_{mn}^2} \vec{\nabla}_{tr} \psi(\vec{s}) \right]$$

<u>Cutoff</u>

$k^2 > 0$ for propagation

$k^2 < 0 \Rightarrow e^{ikz} \to 0$ as $z \to \pm\infty$

(evanescent wave)

$$k_{mn}^2 = \frac{\omega^2}{v^2} - \gamma_{mn}^2 = \frac{\epsilon\mu\omega^2}{c^2} - \gamma_{mn}^2$$

Define $\omega_{mn} \equiv \frac{c}{\sqrt{\mu\epsilon}}\gamma_{mn}$

$$k_{mn}^2 = \frac{\epsilon\mu}{c^2}\left(\omega^2 - \omega_{mn}^2\right)$$

$$k_{mn} = \frac{\sqrt{\epsilon\mu}}{c}\sqrt{\omega^2 - \omega_{mn}^2} = \frac{1}{v}\sqrt{\omega^2 - \omega_{mn}^2}$$

k is real if $\omega > \omega_{mn}$. ω_{mn} is the <u>cutoff frequency</u> for the mn^{th} mode

TE waves transverse electric

$E_z(\vec{r}) = 0$ everywhere, so $E_z\big|_s = 0$ (Dirichlet B.C.) is automatically satisfied

We must also impose the Neumann B.C. $\hat{n}\cdot\vec{\nabla}B_z\big|_s = 0$

$$\vec{E}_{tr}(\vec{r}) = \vec{E}_{tr}(\vec{s})\, e^{ikz}$$

$$\vec{B}_{tr}(\vec{r}) = \vec{B}_{tr}(\vec{s})\, e^{ikz}$$

$$B_z(\vec{r}) = \chi(\vec{s})\, e^{ikz}$$

Helmholtz: $\left(\nabla^2 + \frac{\omega^2}{v^2}\right)B_z(\vec{r}) = 0 \Rightarrow \left(\nabla_{tr}^2 + \gamma^2\right)\chi(\vec{s}) = 0$

$\gamma^2 = \mu\epsilon\frac{\omega^2}{c^2} - k^2$ same as TM

B.C. is $\hat{n}\cdot\vec{\nabla}_{tr}\chi(\vec{s})\big|_s = 0$

$$\vec{\nabla}_{tr}\times\vec{B}_{tr}(\vec{s}) = 0 \Longrightarrow \vec{B}_{tr}(\vec{s}) = -\vec{\nabla}_{tr}\Phi(\vec{s})$$

$$\vec{\nabla}_{tr}\cdot\vec{B}_{tr}(\vec{s}) = -\frac{\partial B_z}{\partial z} = -ik\,\chi(\vec{s})$$

$$\Longrightarrow \nabla_{tr}^2\Phi(\vec{s}) = ik\,\chi(\vec{s})$$

Try $\Phi(\vec{s}) = A\,X(\vec{s}) \implies A = \frac{-ik}{\gamma^2}$

$$\vec{B}_{tr}(\vec{s}) = \frac{ik}{\gamma^2}\vec{\nabla}_{tr}X(\vec{s})$$

$$\vec{\nabla}_{tr}\cancelto{0}{E_z}(\vec{r}) - \frac{\partial}{\partial z}\vec{E}_{tr}(\vec{r}) = \frac{i\omega}{c}\hat{e}_z \times \vec{B}_{tr}(\vec{r})$$

$$\implies -k\vec{E}_{tr}(\vec{s}) = \frac{\omega}{c}\hat{e}_z \times \vec{B}_{tr}(\vec{s})$$

$$\vec{B}_{tr}(\vec{s}) = \frac{\mu}{Z_{TE}}\hat{e}_z \times \vec{E}_{tr}(\vec{s})$$ where $Z_{TE} = \frac{\mu\omega}{kc}$

TE wave impedance

$k_{mn}^2 = \frac{1}{v}\sqrt{\omega^2 - \omega_{mn}^2}$ same as TM modes

---

Phase velocity $v_\varphi = \frac{\omega}{k_{mn}} = \frac{v\omega}{\sqrt{\omega^2-\omega_{mn}^2}} = \frac{v}{\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}} > v$

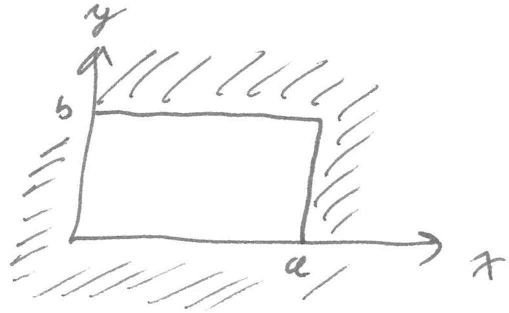
Group velocity $v_g = \frac{\partial\omega}{\partial k_{mn}} = \frac{1}{\frac{\partial k_{mn}}{\partial\omega}} = \frac{1}{\frac{\omega}{v}\frac{1}{\sqrt{\omega^2-\omega_{mn}^2}}}$

$$= v\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} < v$$

where $v = \frac{c}{\sqrt{\mu\epsilon}}$ free space (no boundaries) wave speed (also TEM)

$$v_g \cdot v_\varphi = v^2$$

Example Rectangular Wave guide



$$\vec{s} = \hat{e}_x x + \hat{e}_y y$$

$$\left(\nabla_{tr}^2 + \gamma^2\right)\psi(\vec{s}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)\psi(x,y) = 0$$

TM

$$\psi = E_z = E_{mn}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \Longrightarrow E_z(\vec{s})\Big|_s = 0$$ on boundary

$$\gamma_{mn} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \qquad m, n = 1, 2, 3, \ldots$$

TE $$B_z(\vec{s}) = B_{mn}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

$$\Longrightarrow \hat{n}\cdot\vec{\nabla}_{tr} B_z(\vec{s})\Big|_s = 0$$ on boundary

$$\gamma_{mn} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$ again, but $m, n = 0, 1, 2, 3, \ldots$

but not both $m=0$ <u>and</u> $n=0$ at the same time.

For $a > b$, the lowest cutoff frequency is for the TE wave with $n=1$ and $m=0$.

$$\omega_{01} = \gamma_{01} v = \frac{\pi v}{a}$$

In an infinite medium: $k_\infty = \frac{\omega_{01}}{v} = \frac{\pi}{a}$, $\lambda_\infty = \frac{2\pi}{k_\infty} = 2a$

2/11 <u>Poynting Vector</u> - energy transmitted down waveguide

$$\langle \vec{S} \rangle = \frac{c}{8\pi\mu} \, Re(\vec{E} \times \vec{B}^*)$$

Consider a <u>TM mode</u>

$$\vec{E} = [\vec{E}_{tr}(\vec{\xi}) + \vec{\epsilon}_3 \psi] e^{ikz} = \left[\frac{ik}{\gamma^2} \vec{\nabla}_\xi \psi + \vec{\epsilon}_3 \psi\right] e^{ikz}$$

$$\vec{B} = \frac{\mu}{Z} \vec{\epsilon}_3 \times \vec{E}_{tr}(\vec{\xi}) e^{ikz} \quad \text{with} \quad Z = \frac{kc}{\epsilon\omega}$$

Then $\vec{E} \times \vec{B}^* = \frac{\mu}{Z}[\vec{E}_{tr}(\vec{\xi}) + \vec{\epsilon}_3 \psi] \times [\vec{\epsilon}_3 \times \vec{E}^*_{tr}(\vec{\xi})]$

$$= \frac{\mu}{Z}\left[\vec{\epsilon}_3 \vec{E}_{tr}(\vec{\xi}) \cdot \vec{E}^*_{tr}(\vec{\xi}) - \vec{E}^*_{tr}(\vec{\xi}) \vec{\epsilon}_3 \cdot \vec{E}_{tr}(\vec{\xi}) + (\vec{\epsilon}_3 \vec{\epsilon}_3 \cdot \vec{E}^*_{tr} - \vec{E}^*_{tr})\psi\right]$$

Note $\vec{\epsilon}_3 \cdot \vec{E}_{tr} = 0$, and $\vec{\epsilon}_3$ is real so $\vec{\epsilon}_3 \cdot \vec{E}^*_{tr} = 0$, too.

Then

$$\langle \vec{S} \rangle = \frac{c}{8\pi\mu} \frac{\mu}{Z}\left[\vec{\epsilon}_3 \vec{E}_{tr}(\vec{\xi}) \cdot \vec{E}^*_{tr}(\vec{\xi}) - Re(\underbrace{\psi E^*_{tr}(\vec{\xi})}_{Re(i\psi \vec{\nabla}_\xi \psi^*)})\right]$$

For <u>non-degenerate</u> system, the solutions must be real such that ψ is real with a constant phase, so that the last term vanishes.

For the degenerate case, you can get circulations so this term doesn't necessarily vanish.

But we are interested in power propagating down wave guide. $\langle \vec{S} \rangle$ is the energy per unit area per unit time.

$$\vec{\epsilon}_3 \cdot \langle \vec{S} \rangle = \frac{c}{8\pi Z}\left[\vec{E}_t(\vec{\xi}) \cdot \vec{E}^*_t(\vec{\xi})\right]$$

$$P = \int dA \, \vec{\epsilon}_3 \cdot \langle \vec{S} \rangle = \frac{c}{8\pi Z} \int dA \, \frac{k^2}{\gamma^4} \vec{\nabla}_\xi \psi \cdot \vec{\nabla}_\xi \psi^*$$

Green's identity in two dimensions says

$$\int_A dA \, (\Phi_1 \nabla_\xi^2 \Phi_2 + \vec{\nabla}_\xi \Phi_1 \cdot \vec{\nabla}_\xi \Phi_2) = \oint_C dl \, \Phi_1 \vec{n} \cdot \vec{\nabla}_\xi \Phi_2$$

$$\therefore \int dA \, \vec{\nabla}_\xi \psi^* \cdot \vec{\nabla}_\xi \psi = \oint_C dl \, \psi^*(\vec{\xi}) \vec{n} \cdot \vec{\nabla}_\xi \psi - \int_A dA \, \psi^*(\vec{\xi}) \nabla^2 \psi(\vec{\xi})$$

But the B.C. for TM is $\Psi(\vec{\xi}) = 0$ on C

From the Helmholtz equation

$$\nabla_\xi^2 \Psi = -\gamma^2 \Psi$$

So $$P = \frac{c}{8\pi Z} \frac{k^2}{\gamma^2} \int dA\, \Psi^*(\vec{\xi}) \Psi(\vec{\xi})$$

Next look at average energy density

$$\langle u \rangle = \frac{1}{16\pi} \left(\epsilon \vec{E}^* \cdot \vec{E} + \frac{1}{\mu} \vec{B}^* \cdot \vec{B} \right)$$

$$\left(u = \frac{1}{4\pi} \left(E^2 \epsilon + \frac{1}{\mu} B^2 \right) \right)$$

Put in the expression for $\vec{E}$ from above and

$$\vec{B}_{tr} = \frac{\mu}{Z} \vec{e}_3 \times \vec{E}_{tr} e^{ikz}$$

$$\langle u \rangle = \frac{1}{16\pi} \left[\epsilon \vec{E}_{tr}^*(\vec{\xi}) \cdot \vec{E}_{tr}(\vec{\xi}) + \frac{\mu}{Z^2} (\vec{e}_3 \times \vec{E}_{tr}^*(\vec{\xi})) \cdot (\vec{e}_3 \times \vec{E}_{tr}(\vec{\xi})) + \epsilon \Psi^*(\vec{\xi}) \Psi(\vec{\xi}) \right]$$

$$= \frac{1}{16\pi} \left[\left(\epsilon + \frac{\epsilon^2 \mu \omega^2}{k^2 c^2} \right) \vec{E}_{tr}^* \cdot \vec{E}_{tr}(\vec{\xi}) + \epsilon \Psi^*(\vec{\xi}) \Psi(\vec{\xi}) \right]$$

Integrate over cross section to get energy per unit length.

$$U' = \int dA \langle u \rangle$$

$$= \frac{\epsilon}{16\pi} \left[\left(1 - \frac{\omega^2}{k^2 v^2} \right) \int dA \frac{k^2}{\gamma^4} \vec{\nabla}\Psi^*(\vec{\xi}) \cdot \vec{\nabla}\Psi(\vec{\xi}) + \int dA\, \Psi^* \Psi(\vec{\xi}) \right]$$

Use Green's identity again to get

$$U' = \frac{\epsilon}{16\pi} \left[\left(1 + \frac{\omega^2}{k^2 v^2} \right) \frac{k^2}{\gamma^2} + 1 \right] \int dA\, \Psi^*(\vec{\xi}) \Psi(\vec{\xi})$$

But $k^2 = \frac{\omega^2}{v^2} - \gamma^2 = \frac{\omega^2 - \omega_\lambda^2}{v^2}$ so

$$\left(1 + \frac{\omega^2}{k^2 v^2} \right) \frac{k^2}{\gamma^2} + 1 = \left(1 + \frac{k^2 + \gamma^2}{k^2} \right) \frac{k^2}{\gamma^2} + 1 = 2 \left(\frac{k^2}{\gamma^2} + 1 \right)$$

$$= \frac{2\omega^2}{\gamma^2 v^2}$$ and

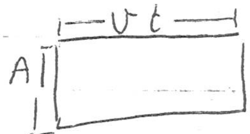
$$U' = \frac{\epsilon}{8\pi} \frac{\omega^2}{\gamma^2 v^2} \int dA\, \Psi^*(\vec{\xi}) \Psi(\vec{\xi})$$

Then $\dfrac{P}{U'} = \dfrac{\frac{c}{8\pi Z} \frac{k^2}{\gamma^2} \int dA\, \Psi^*(\vec{\xi})\, \Psi(\vec{\xi})}{\frac{\epsilon}{8\pi} \frac{\omega^2}{\gamma^2 v^{-2}} \int dA\, \Psi^*(\vec{\xi}) \Psi(\vec{\xi})}$

$$= \frac{c}{Z\epsilon} \frac{k^2 v^{-2}}{\omega^2} = v^2 \frac{k}{\omega} = v\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_g$$

$\therefore P = U' v_g$

and energy flows down the guide with the group velocity v_g. The same result holds for TE wave.
Could also see roughly from dimensional considerations



v = rate of flow of energy

$\dfrac{A v t \langle u \rangle}{t} = \dfrac{v t U'}{t} = P$

energy in the volume (Avt).

Standing Waves; Resonant Cavities

Put conductors on the ends of the wave guide creating a resonant cavity in which there are electromagnetic standing waves. We can have only $\omega_{\lambda p}$ (p = mode width) instead of $\omega > \omega_\lambda$. P is determined by the new boundaries and subsequent boundary conditions: $\vec{n} \times \vec{E} = 0$, $\vec{n} \cdot \vec{B} = 0$.

TM modes

$B_z = 0$ so $\vec{n} \cdot \vec{B} = 0$ on ends is satisfied

We need

$$\vec{\nabla}_{tr} \times \vec{E}_{tr} = 0 \qquad \vec{\nabla}_{tr} \cdot \vec{E}_{tr} = -\frac{\partial E_z}{\partial z}$$

$$\left(\nabla^2 + \frac{\omega^2}{v^{-2}}\right) E_z = 0 \qquad -\frac{\partial \vec{B}_{tr}}{\partial z} = -i \mu\epsilon \frac{\omega}{c} \vec{e}_3 \times \vec{E}_{tr}$$

Set $E_z = \Psi(\vec{\xi})\, \zeta(z)$

Now the Helmholz equation becomes

$$\frac{\left(\nabla_\xi^2 + \frac{\omega^2}{v^{-2}}\right)\Psi(\vec{\xi})}{\Psi(\vec{\xi})} + \frac{\frac{d^2}{dz^2}\zeta(z)}{\zeta(z)} = 0$$

Then $\dfrac{d^2}{dz^2}\zeta(z) + k^2 \zeta(z) = 0$

$$(\nabla_\xi^2 + \gamma^2)\psi(\vec{\xi}) = 0 \quad \text{with } \gamma^2 = \frac{\omega^2}{v^2} - k^2$$

Set $\vec{E}_{tr} = A\, \vec{\nabla}_\xi \psi(\vec{\xi}) \frac{d}{dz} \varsigma(z)$

Choose A to satisfy above conditions $\vec{\nabla}_\xi \times \vec{E}_{tr} = 0$ and $\vec{\nabla}_\xi \cdot \vec{E}_{tr} = -\frac{\partial E_z}{\partial z}$

$$A\, \underset{-\gamma^2\psi(\vec{\xi}) \text{ from Helmholtz}}{\underbrace{\nabla_\xi^2 \psi(\vec{\xi})}} \frac{d}{dz}\varsigma(z) = -\psi(\vec{\xi}) \frac{d}{dz}\varsigma(z)$$

$$\therefore\ A = \frac{1}{\gamma^2}$$

$$\vec{E}_{tr} = \frac{1}{\gamma^2} \vec{\nabla}_\xi \psi(\vec{\xi}) \frac{d}{dz}\varsigma(z)$$

To make $\vec{E}_{tr} = 0$ at $z = 0$ and $z = d$ choose

$$\varsigma(z) = \cos kz \qquad k = \frac{\pi p}{d}, P = 0, 1, 2.$$

$$\therefore\ \vec{E}_{tr} = -\frac{k^2}{\gamma^2} \vec{\nabla}_\xi \psi(\vec{\xi}) \sin \frac{\pi p z}{d}$$

$$E_z = \psi(\vec{\xi}) \cos \frac{\pi p z}{d}$$

Then $-\frac{\partial \vec{B}_{tr}}{\partial z} = -i\frac{\mu\epsilon\omega}{c} \vec{\epsilon}_3 \times \vec{E}_{tr}$

$$\frac{\partial \vec{B}_{tr}}{\partial z} = \frac{i\mu\epsilon\omega}{c} \vec{\epsilon}_3 \times \vec{E}_{tr}$$

$$\boxed{\vec{B}_{tr} = \frac{i\mu\epsilon\omega}{\gamma^2 c} \vec{\epsilon}_3 \times \vec{\nabla}_\xi \psi(\vec{\xi}) \cos\left(\frac{\pi p z}{d}\right)}$$

Note: for $p = 0$, $\vec{E}_{tr} = 0$, so the electric field is longitudinal and the magnetic field is transverse.

Eigenfrequencies come from

$$(\vec{\nabla}_\xi^2 + \gamma^2)\psi(\vec{\xi}) = 0 \qquad \psi(\vec{\xi}) = 0 \text{ on lateral surfaces)}$$

which gives the same values as thee wave guide

$$\gamma_\lambda^2 = \frac{\omega^2}{v^2} - k^2 = \frac{\omega^2}{v^2} - \frac{\pi^2 p^2}{d^2}$$

$$\omega = \omega_{\lambda,p} = v\sqrt{\gamma_\lambda^2 + \frac{\pi^2 p^2}{d^2}}$$

TE mode is more of the same.