Now suppose we did experiments to determine $\underline{\underline{\rho}}$

Since ρ_{ii} are real, $\underline{\underline{\rho}}$ must be of the form

$$\underline{\underline{\rho}} = \begin{pmatrix} a & b+ic \\ b-ic & 1-a \end{pmatrix}$$

$\underline{\underline{\rho}}^{\dagger} = \underline{\underline{\rho}}$ hermitian

$\mathrm{Tr}(\underline{\underline{\rho}}) = 1$

so we need to perform 3 experiments. Suppose the results of the experiments imply that $\mathrm{Tr}(\underline{\underline{\rho}}^2) = \mathrm{Tr}(\underline{\underline{\rho}})$. We now show that this condition implies that the radiation is polarized. To show this, we note that since $\underline{\underline{\rho}}$ is hermitian it can be brought into diagonal form by a unitary transformation. If the eigenvalues of $\underline{\underline{\rho}}$ are distinct (non-degenerate) then the normalized eigenvectors must be <u>orthogonal</u>

$$\underline{\underline{u}}^{\dagger}_{(i)}\, \underline{\underline{u}}_{(j)} = \delta_{ij} \qquad (i,j = 1,2)$$

$\underline{\underline{u}}^{\dagger}$ row vector

$\underline{\underline{u}}$ column vector

dot (scalar) product

The eigenvalues are defined as ρ_i with

$$\underline{\underline{\rho}}\, \underline{\underline{u}}_{(i)} = \rho_i\, \underline{\underline{u}}_{(i)}$$

Since the eigenvectors are <u>complete</u>, we have

$$\sum_{i=1}^{2} \underline{\underline{u}}_{(i)}\, \underline{\underline{u}}^{\dagger}_{(i)} = \underline{\underline{\mathbb{I}}}_2 = \text{2x2 identity matrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

outer product

Then right multiplying the eigenvector equation by $\underline{\underline{u}}^{\dagger}_{(i)}$ we get

$$\underline{\underline{\rho}}\,\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)} = \rho_i\,\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)}$$ now sum over i

$$\underline{\underline{\rho}}\sum_{i=1}^{2}\underline{\underline{u}}_{(i)}\underline{\underline{u}}^{\dagger}_{(i)} = \underline{\underline{\rho}}\,\underline{\underline{I}}_2 = \underline{\underline{\rho}} = \sum_{i=1}^{2}\rho_i\,\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)}$$

↑ spectral decomposition

$$\underline{\underline{\rho}}^2 = \left(\sum_{i=1}^{2}\rho_i\,\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)}\right)\left(\sum_{j=1}^{2}\rho_j\,\underline{\underline{u}}_{(j)}\,u^{\dagger}_{(j)}\right)$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}\rho_i\,\rho_j\,\underline{\underline{u}}_{(i)}\,\underbrace{\underline{\underline{u}}^{\dagger}_{(i)}\,\underline{\underline{u}}_{(j)}}_{\delta_{ij}}\,\underline{\underline{u}}^{\dagger}_{(j)}$$

$$= \sum_{i=1}^{2}\rho_i^2\,\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)}$$ In fact $\underline{\underline{\rho}}^n = \sum_{i=1}^{2}\rho_i^n\,\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)}$

Note also $\mathrm{Tr}\left[\underline{\underline{u}}_{(i)}\,\underline{\underline{u}}^{\dagger}_{(i)}\right] = \mathrm{Tr}\left[\underline{\underline{u}}^{\dagger}_{(i)}\,\underline{\underline{u}}_{(i)}\right] = \underline{\underline{u}}^{\dagger}_{(i)}\,\underline{\underline{u}}_{(i)} = 1$

↑ scalar product

So $\mathrm{Tr}(\underline{\underline{\rho}}^2) = \mathrm{Tr}(\underline{\underline{\rho}})$ implies

$$\sum_{i=1}^{2}\rho_i^2 = \sum_{i=1}^{2}\rho_i$$

or $\sum_i(\rho_i^2 - \rho_i) = 0$ or $\sum_i \rho_i(1-\rho_i) = 0$

or $\rho_1(1-\rho_1) + \rho_2(1-\rho_2) = 0$

Since $0 \le \rho_i \le 1$ then each term is positive or zero and hence each term <u>must</u> be zero.

Thus $\rho_1(1-\rho_1) = 0$ <u>and</u> $\rho_2(1-\rho_2) = 0$

But the sum $\rho_1 + \rho_2$ must be 1 so

either $\rho_1 = 1$ and $\rho_2 = 0$ <u>or</u> $\rho_1 = 0$ and $\rho_2 = 1$.

This means $\underline{\underline{\rho}} = \underline{\underline{u}}_{(1)} \underline{\underline{u}}_{(1)}^{+}$ or $\underline{\underline{\rho}} = \underline{\underline{u}}_{(2)} \underline{\underline{u}}_{(2)}^{+}$

But this is exactly the form of a polarized wave with $\underline{\underline{u}} = \dfrac{\underline{\underline{E}}_0}{\sqrt{\underline{\underline{E}}_0^{+} \underline{\underline{E}}_0}}$.

Polarized radiation $\Longleftrightarrow$ $Tr(\underline{\underline{\rho}}^2) = 1$

<u>or</u>

Polarized radiation $\Longleftrightarrow$ $Tr(\underline{\underline{J}}^2) = [Tr(\underline{\underline{J}})]^2$

If the radiation is <u>not</u> polarized, then

$$\underline{\underline{\rho}} = \rho_1 \, \underline{\underline{u}}_{(1)} \underline{\underline{u}}_{(1)}^{\dagger} + \rho_2 \, \underline{\underline{u}}_{(2)} \underline{\underline{u}}_{(2)}^{\dagger}$$

If $\rho_1 \neq \rho_2$ (eigenvalues are not degenerate) then $\rho_2 = 1 - \rho_1$ and $\underline{\underline{u}}_{(i)}$ is unique up to a phase factor. Since $\underline{\underline{\rho}}$ must contain 3 independent parameters the eigenvectors $\underline{\underline{u}}_{(i)}$ must contain 2 independent parameters. The phase does not count as an independent parameter because it contributes nothing to $\underline{\underline{\rho}}$.

To parametrize $\underline{\underline{u}}_{(i)}$ write $\underline{\underline{u}}_{(i)} = \underline{\underline{R}} \, \underline{\underline{v}}_{(i)}$

where $\underline{\underline{R}}$ is the 2x2 real orthogonal matrix

$$\underline{\underline{R}} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \qquad \text{and}$$

$$\underline{\underline{v}}_{(1)} = \frac{1}{\sqrt{2-\epsilon^2}} \begin{pmatrix} 1 \\ i\sqrt{1-\epsilon^2} \end{pmatrix} , \quad \underline{\underline{v}}_{(2)} = \frac{1}{\sqrt{2-\epsilon^2}} \begin{pmatrix} i\sqrt{1-\epsilon^2} \\ 1 \end{pmatrix}$$

$\underline{\underline{R}}$ generates a rotation about the direction of propagation $\hat{k}$ through an angle ψ counterclockwise.

The vectors $\underline{\underline{v}}_{(1)}$ and $\underline{\underline{v}}_{(2)}$ describe ellipses. To see this, define

$$e^{i\varphi_{(i)}}\,\underline{\underline{v}}_{(i)} = \underline{\underline{x}}_{(i)} + i\,\underline{\underline{y}}_{(i)}$$

with $\underline{\underline{x}}_{(i)} = \begin{pmatrix} x_{(i)1} \\ x_{(i)2} \end{pmatrix}$ and $\underline{\underline{y}}_{(i)} = \begin{pmatrix} y_{(i)1} \\ y_{(i)2} \end{pmatrix}$ <u>real</u>

If, for general phase angle $\varphi_{(i)}$, we put in the expressions above for $\underline{\underline{v}}_{(i)}$ we will find curves in the $x_{(i)1} - x_{(i)2}$ plane (and also in the $y_{(i)1} - y_{(i)2}$ plane but these give nothing new). The curves turn out to be <u>ellipses</u>.

<u>Homework</u>

Show that the column vector $\underline{\underline{v}}_{(1)}$ describes an ellipse with major axis along $\underline{\underline{\epsilon}}_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, minor axis along $\underline{\underline{\epsilon}}_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and eccentricity ϵ.

Show that $\underline{\underline{v}}_{(2)}$ has major axis along $\underline{\underline{\epsilon}}_{(2)}$, minor axis along $\underline{\underline{\epsilon}}_{(1)}$, and eccentricity ϵ.

Show that $\underline{\underline{v}}_{(i)}^{\dagger}\,\underline{\underline{v}}_{(j)} = \delta_{ij}$ and that $\underline{\underline{R}}^{\dagger}\underline{\underline{R}} = \mathbb{I}_2$ and hence that $\underline{\underline{u}}_{(i)}^{\dagger}\,\underline{\underline{u}}_{(j)} = \delta_{ij}$. What geometric figures do the vectors $\underline{\underline{u}}_{(i)}$ describe?