Consider an initially _linearly_ polarized incident plane wave with polarization vector

$$\vec{\epsilon} = \cos\varphi\,\vec{\epsilon_1} + \sin\varphi\,\vec{\epsilon_2}$$

$$\vec{E}(r,t) = \vec{\epsilon}\,E_0\,\exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

So the incident amplitudes are

$$E_{01} = E_0\cos\varphi \qquad \text{and} \qquad E_{02} = E_0\sin\varphi$$

the reflected wave is then

$$\vec{E}''(r,t) = E_0\left[\cos\varphi\,e^{i\psi_1}\vec{\epsilon_1} + \sin\varphi\,e^{i\psi_2}\vec{\epsilon_2}\right]e^{i(\vec{k}''\cdot\vec{r} - \omega t)}$$
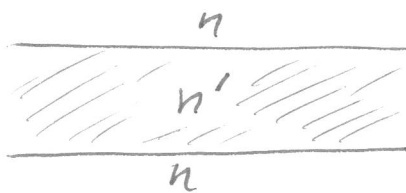
which is an _elliptically_ polarized wave.

In the analysis above we chose

$$\cos(\theta_b) = +\,i\,\sqrt{\frac{n'^2}{n^2}\sin^2(\theta_i) - 1}$$

That was correct for a refracted medium which occupies an infinite half space — that is, we know physically that we should have an exponentially _decaying_ wave as we look further into the medium away from the interface. This is called an evanescent _wave_.

But consider the problem where the medium with index of refraction n' is bounded on both sides by index n:

$$\begin{array}{c} n \\ \hline \text{///////} n' \text{///////} \\ \hline n \end{array}$$

In this case, we will need both an exponentially decaying wave _and_ an exponentially growing wave in the n' medium in order to match boundary conditions at both interfaces. There will be a propagating wave in both n media (with diminished amplitude in the upper one if the wave is incident from the lower one). This situation is analogous to quantum mechanical tunneling and is called frustrated total internal reflection.

Absorption and Dispersion

Model for the Dielectric Function

We give a classical model for the dielectric function $\epsilon(\omega)$. We assume that all charged oscillators are harmonically bound in atoms, Z per atom. We assume that the oscillators are damped to take into account the loss of energy when the atoms radiate. This is not a bad model since even classically the system will have discrete frequencies, just like a quantum system. (The damped oscillators do not change their frequency when the amplitude decreases.) However, the radiation in the model contains harmonics and this is not observed experimentally, so the model isn't __too__ good. We assume that there are many modes of vibration characterized by frequencies ω_n and damping rates γ_n. The coordinates of an electron in the n'th mode of the $a^{\underline{th}}$ atom is $\vec{r}_{an}$ relative to the center of mass of the $a^{\underline{th}}$ atom which is $\vec{R}_a$. We then have

$$m\, \ddot{\vec{r}}_{an}^{\,(t)} + m\gamma_n \dot{\vec{r}}_{an}^{\,(t)} + m\omega_n^2\, \vec{r}_{an}^{\,(t)} = -e\, \text{Re}\left[\vec{E}_0\, e^{i[\vec{k}\cdot(\vec{r}_{an}^{(t)} + \vec{R}_{an}) - \omega t]}\right]$$

$\underset{\text{charge}}{\uparrow}$

for the dynamics of an electron in the presence
of an electromagnetic plane wave (we neglect the
magnetic force which is $O(\frac{v}{c})$ smaller than the
electric force). This differential equation is
non-linear and hence insoluble because of the
$\vec{k} \cdot \vec{r}_{an}$ factor in the exponential. Thus we will
assume that we have sufficiently long wavelengths
so that $\vec{k} \cdot \vec{r}_{an} \ll 1$ and we may neglect $\vec{k} \cdot \vec{r}_{an}$
in the exponential. We then introduce a
complex position vector $\vec{Z}_{an}$ which obeys the
equation of motion

$$\ddot{\vec{Z}}_{an}(t) + \gamma_n \dot{\vec{Z}}_{an}(t) + \omega_n^2 \vec{Z}_{an}(t) = -\frac{e}{m} \vec{E}_0 e^{i(\vec{k} \cdot \vec{R}_a - \omega t)}$$

with $\vec{r}_{an} = \text{Re}[\vec{Z}_{an}]$. We look for forced solutions

$$\vec{Z}_{an}(t) = \vec{Z}_{an}^0 e^{i(\vec{k} \cdot \vec{R}_a - \omega t)}$$

$$\vec{Z}_{an}^0 = \frac{-\frac{e}{m} \vec{E}_0}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$

the dipole moment of the ath atom is then

$$\vec{\mu}_a = \sum_n (-e) f_n \vec{r}_{an} = Re\left[\sum_n (-e) f_n \vec{z}_{an} \right]$$

where f_n = the number of electrons in the nth mode.

$$\vec{\mu}_a = Re\left[\frac{e^2}{m} \sum_n \frac{f_n \vec{E}_0 e^{i(\vec{k}\cdot\vec{R}_a - \omega t)}}{\omega_n^2 - \omega^2 - i\gamma_n \omega} \right]$$

Now consider a volume element of dimension l with $kl \ll 1$. Then the factor $e^{i\vec{k}\cdot\vec{R}_a}$ is essentially the same for all atoms in the element. The polarization is then $\vec{P} = Re[\vec{P}]$ where

$$\vec{P} = \vec{P}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \text{with} \qquad \vec{P}_0 = \frac{Ne^2}{m} \sum_n \frac{f_n \vec{E}_0}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$

N = number of atoms per unit volume.
$\vec{r}$ = location of the center of the volume element.

Now $\quad \vec{D}_0 = \epsilon_0 \vec{E}_0 + \vec{P}_0 \equiv \epsilon(\omega) \vec{E}_0$

with $\quad \epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$

For $\omega \ll (\omega_n)_{min}$ we have

$$\epsilon(\omega) \approx \epsilon(0) = \epsilon_0 + \frac{Ne^2}{m} \sum_n \frac{f_n}{\omega_n^2}$$

If $(\omega_n)_{max}$ is the highest <u>occupied</u> frequency then

for $\omega \gg (\omega_n)_{max}$ we have

$$\epsilon(\omega) \approx \epsilon_0 - \frac{Ne^2}{m} \sum_n \frac{f_n}{\omega^2} = \epsilon_0 - \frac{Ne^2 Z}{m \omega^2}$$

where we used $\sum_n f_n = Z_1 =$ number of electrons per atom

The expression above for the dielectric function also arises in a quantum mechanical treatment provided we make the following interpretation:

(a) $\omega_n = \dfrac{E_n - E_0}{\hbar}$ where $E_n =$ energy of the nth atomic energy level and $E_0 =$ ground state energy. This assumes the atom is in its ground state (we neglect thermal excitation.) The sums in the formulae above then go over only the excited states.

(b) $\gamma_n = \dfrac{1}{\tau_n}$ where τ_n is the lifetime of the excited state of the atom. The excited state emits radiation spontaneously and does not live indefinitely.

(c) $f_n =$ "oscillator strength" which involves matrix elements of the electric dipole moment operator. The result $\sum_n f_n = Z$ still holds and is called the "f-sum rule".

We recall that the connection between the wave number k and frequency ω is

$$k^2 = \frac{\omega^2}{v^2} = \omega^2 \, \epsilon(\omega) \qquad \text{where we take } \mu(\omega) \approx \mu_0$$
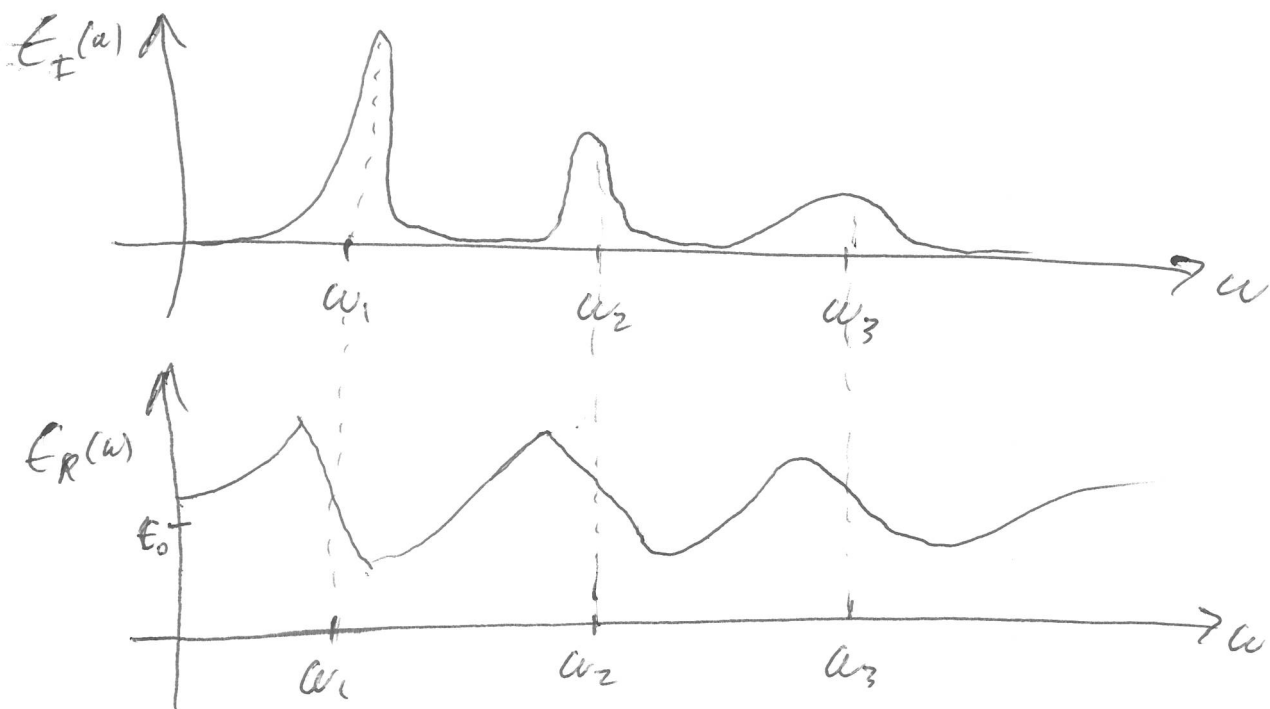
Since $\epsilon(\omega)$ is generally complex, then so is k^2 and so is k. We write

$$\epsilon(\omega) = \epsilon_R(\omega) + i\, \epsilon_I(\omega)$$

$$k = \beta + \frac{i}{2}\alpha$$

$$\beta^2 - \frac{\alpha^2}{4} = \epsilon_R(\omega)\, \omega^2$$

$$\alpha\beta = \epsilon_I(\omega)\, \omega^2$$

$$\epsilon_R(\omega) = \epsilon_0 + \frac{Ne^2}{m} \sum_n \frac{f_n(\omega_n^2 - \omega^2)}{(\omega^2 - \omega_n^2)^2 + \omega^2 \gamma_n^2}$$

$$\epsilon_I(\omega) = \frac{Ne^2}{m} \sum_n \frac{f_n \gamma_n \omega}{(\omega^2 - \omega_n^2)^2 + \omega^2 \gamma_n^2}$$

The imaginary part of $\epsilon(\omega)$ is sizeable only near a resonance $\omega \approx \omega_n$. The peak value of $\epsilon_I(\omega)$ diminishes as the resonance frequency increases and the peaks get broader because the higher excitations are increasingly more unstable — smaller lifetime τ_n therefore larger γ_n.

Now suppose $\epsilon_I \ll \epsilon_R$. Then we can find α and β approximately as

$$\beta^2 - \frac{\alpha^2}{4} \approx \beta^2 = \epsilon_R(\omega)\,\omega^2 \implies \beta = \sqrt{\epsilon_R(\omega)}\;\omega$$

$$\alpha = \frac{1}{\beta}\epsilon_I(\omega)\,\omega^2 \implies \alpha = \frac{\epsilon_I(\omega)}{\sqrt{\epsilon_R(\omega)}}\,\omega = \frac{\epsilon_I(\omega)}{\epsilon_R(\omega)}\beta$$

If the plane wave is propagating in the z-direction

$$\vec{\mathcal{E}}(z,t) = R_e\left[\vec{E}_0\, e^{i(kz-\omega t)}\right] = R_e\left[\vec{E}_0\, e^{i(\beta z - \omega t)}\right]e^{-\frac{\alpha z}{2}}$$

Thus as the wave propagates, the amplitude attenuates. Furthermore, the phase velocity is

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\epsilon_R(\omega)}} = \frac{c}{n(\omega)}$$

where $n(\omega) = c\sqrt{\epsilon_R(\omega)}$ is the index of refraction.

Usually $n(\omega)$ is an <u>increasing</u> function of ω.
(Blue light refracts more than red light)

$\downarrow$ red

$\uparrow$ blue

Rainbow: