To see the significance of the labels "retarded" and "advanced" we calculate (with $T \equiv t-t'$)

$$G^r(R;T) = \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} G^r_\omega(R) e^{-i\omega T} = \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(T-\frac{R}{c})}}{R}$$

$$= \frac{\delta(T-\frac{R}{c})}{R}$$

or

$$G^r(\vec{r}-\vec{r}\,'; t-t') = \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}$$

Note that $G^r(\vec{r}-\vec{r}\,'; t-t') = 0$ unless $t = t' + \frac{|\vec{r}-\vec{r}\,'|}{c}$.

This means that an instantaneous source disturbance at $\vec{r}\,'$ and t' will be experienced at the field point $\vec{r}$ only at the <u>later</u> time $t = t' + \frac{|\vec{r}-\vec{r}\,'|}{c}$. The signal at $\vec{r}$ is felt only after the transit time $\frac{|\vec{r}-\vec{r}\,'|}{c}$; that is, the signal is retarded or delayed by that amount of time. In the following, we will use G^r because of the causality aspect. There are occasions when G^a is needed. In quantum field theory the half advanced / half retarded Feynman Green function or "propagator" is used.

We also restrict ourselves to the Coulomb Gauge.

Under Fourier time transform we have:

$$\vec{\tilde{E}}_\omega(\vec{r}) = -\vec{\nabla}\tilde{\Phi}_\omega(\vec{r}) + \frac{i\omega}{c}\vec{\tilde{A}}_\omega(\vec{r})$$

$$\vec{\tilde{B}}_\omega(\vec{r}) = \vec{\nabla}\times\vec{\tilde{A}}_\omega(\vec{r})$$

where $\frac{\partial}{\partial t} \rightarrow -i\omega$ under the time transform

---

Also $$\tilde{\Phi}_\omega(\vec{r}) = \iiint dV' \frac{\tilde{\rho}_\omega(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

$G_0(\vec{r}-\vec{r}')$ Poisson Green function (no time dependence)

$$\vec{\tilde{A}}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\iiint dV' \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\left[\vec{\tilde{J}}_\omega(\vec{r}')\right]_{tr}$$

$\tilde{G}^r_\omega(\vec{r}-\vec{r}')$ Fourier time transform of the retarded Green function for the wave equation.

We only want to calculate the fields far from the sources which are assumed to be localized.

Using $\frac{1}{|\vec{r}-\vec{r}'|} \rightarrow \frac{1}{r}$ in the limit, we obtain

$$\tilde{\Phi}_\omega(\vec{r}) \rightarrow \frac{\tilde{Q}_\omega}{4\pi\epsilon_0 r}$$

where $$\tilde{Q}_\omega = \iiint \tilde{\rho}_\omega(\vec{r}')\,dV' = \int_{t=-\infty}^{\infty} e^{i\omega t} Q(t)\,dt$$

but Q the electric charge is a constant so

$$\tilde{Q}_\omega = Q\int_{t=-\infty}^{\infty} e^{i\omega t}\,dt = Q\,2\pi\,\delta(\omega)$$

For $\tilde{G}^r_\omega(\vec{r}-\vec{r}\,')$ we first note that

$$|\vec{r}-\vec{r}\,'| = \sqrt{r^2 - 2rr'\cos\gamma + r'^2} = r\sqrt{1 - \frac{2r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2}$$

$$\rightarrow r\left[1 - \frac{r'}{r}\cos\gamma + O\!\left(\frac{1}{r^2}\right)\right] = r - r'\cos\gamma + O\!\left(\frac{1}{r}\right)$$

so $$\tilde{G}^r_\omega(\vec{r}-\vec{r}\,') \rightarrow \frac{e^{i\frac{\omega}{c}r}}{r}\, e^{-i\vec{k}\cdot\vec{r}\,'}$$

where $\vec{k} = \frac{\omega}{c}\hat{n}$ and $\hat{n} \equiv \frac{\vec{r}}{r}$. Then

$$\tilde{\vec{A}}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\iiint dV'\, e^{-i\vec{k}\cdot\vec{r}\,'}\left[\tilde{\vec{J}}_\omega(\vec{r}\,')\right]_{tr}$$

$$\tilde{\vec{A}}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\left[\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right]_{tr} = \frac{\mu_0}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\left[\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} - \hat{n}\left(\hat{n}\cdot\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)\right]$$

where we used $$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} \equiv \iiint dV'\, e^{-i\vec{k}\cdot\vec{r}\,'}\,\tilde{\vec{J}}_\omega(\vec{r}\,')$$

We note that in the far field ($r \to \infty$)

$\tilde{\Phi}_\omega(\vec{r}) \longrightarrow \dfrac{2\pi\,\delta(\omega)\,Q}{r}$ and the electric field

from $-\vec{\nabla}\tilde{\Phi}_\omega(\vec{r})$ falls like $\frac{1}{r^2}$. We are

interested only in radiation fields which fall

off like $\frac{1}{r}$. Both the electric and magnetic

fields come from the vector potential $\tilde{\vec{A}}_\omega(\vec{r})$

in Coulomb gauge (Radiation gauge).

$$\tilde{\vec{E}}_\omega(\vec{r}) \longrightarrow i\omega\,\frac{\mu_0}{4\pi}\,\frac{e^{i\frac{\omega}{c}r}}{r}\left[\tilde{\vec{J}}_{\vec{k}\omega} - \hat{n}\,\hat{n}\cdot\tilde{\vec{J}}_{\vec{k}\omega}\right]$$

The magnetic field is obtained from $\tilde{\vec{B}}_\omega(\vec{r}) = \vec{\nabla}\times\tilde{\vec{A}}_\omega(\vec{r})$

which in the far field comes from $\vec{\nabla} e^{i\frac{\omega}{c}r}$ only.

$$\tilde{\vec{B}}_\omega(\vec{r}) \longrightarrow \frac{i\omega}{c}\,\frac{\mu_0}{4\pi}\,\frac{e^{i\frac{\omega}{c}r}}{r}\;\hat{n}\times\left[\tilde{\vec{J}}_{\vec{k}\omega} - \hat{n}\,\hat{n}\cdot\tilde{\vec{J}}_{\vec{k}\omega}\right]$$

$$= \frac{\hat{n}}{c}\times\tilde{\vec{E}}_\omega(\vec{r})$$

where we used $\vec{\nabla} r = \frac{\vec{r}}{r} = \hat{n}$

Thus we find that $\tilde{\vec{E}}_\omega(\vec{r})$ and $\tilde{\vec{B}}_\omega(\vec{r})$ are each perpendicular to $\hat{n}$ and to one another.

$\hat{n}$, $\tilde{\vec{E}}_\omega$, and $\tilde{\vec{B}}_\omega$ form a triad in the far field:

$$\hat{n}\cdot\tilde{\vec{E}}_\omega = 0 = \hat{n}\cdot\tilde{\vec{B}}_\omega \quad ; \quad \tilde{\vec{E}}_\omega = c\,\tilde{\vec{B}}_\omega \times \hat{n} \quad ; \quad c\tilde{\vec{B}}_\omega = \hat{n}\times\tilde{\vec{E}}_\omega$$

Bear in mind that there are <u>near field</u> contributions which fall off faster than $\frac{1}{r}$ and these do not generally have this triad behavior.

A quantity of interest is the energy radiated to regions remote from the source. Consider first the case in which the source is turned on only for a finite amount of time. Then we can calculate the total amount of energy radiated through a large sphere S of radius r.

real Poynting vector

$$W = \int_{-\infty}^{\infty} dt \oint_S dS\, \vec{\mathcal{S}}(\vec{r},t) \cdot \hat{n}$$

solid angle   real fields

$$= \frac{1}{\mu_0} \int_{-\infty}^{\infty} dt \int d\Omega\, r^2\, \hat{n} \cdot \vec{\mathcal{E}}(\vec{r},t) \times \vec{\mathcal{B}}(\vec{r},t)$$

Since we will consider the limit $r \to \infty$, we need only the $\mathcal{O}(\frac{1}{r})$ contributions from the fields.

Substituting the time Fourier transforms, we have

$$W = \frac{1}{\mu_0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} dt \int d\Omega\, r^2\, \hat{n} \cdot \tilde{\vec{E}}_\omega(\vec{r}) \times \tilde{\vec{B}}_{\omega'}(\vec{r})\, e^{-i(\omega+\omega')t}$$

and $\displaystyle\int_{-\infty}^{\infty} dt\, e^{-i(\omega+\omega')t} = 2\pi\, \delta(\omega+\omega')$ so

$$W = \frac{1}{\mu_0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d\Omega\, r^2\, \hat{n} \cdot \tilde{\vec{E}}_\omega(\vec{r}) \times \tilde{\vec{B}}^{*}_\omega(\vec{r})$$

where we used $\tilde{\vec{B}}_{-\omega}(\vec{r}) = \tilde{\vec{B}}^{*}_\omega(\vec{r})$

$$= \frac{1}{\mu_0 c} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d\Omega\, r^2\, \tilde{\vec{E}}^{*}_\omega(\vec{r}) \cdot \tilde{\vec{E}}_\omega(\vec{r}) \equiv \int_{\omega=0}^{\infty} d\omega \int d\Omega\, W(\omega,\Omega)$$

$W(\omega, \Omega)$ is the energy radiated in a frequency interval $d\omega$ centered on ω, into a solid angle $d\Omega$ centered on Ω.

$$W(\omega, \Omega) = \frac{1}{\mu_0 c \pi} r^2 \tilde{\tilde{\vec{E}}}_\omega^*(\vec{r}) \cdot \tilde{\tilde{\vec{E}}}_\omega(\vec{r})$$

$$= \frac{1}{\mu_0 c \pi} \omega^2 \left(\frac{\mu_0}{4\pi}\right)^2 \left[\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} - \hat{n}\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right]^* \cdot \left[\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} - \hat{n}\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right]$$

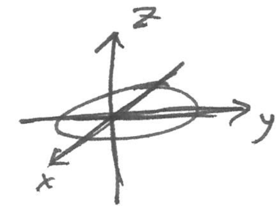
$$= \frac{\mu_0 \omega^2}{16\pi^3 c} \left[\tilde{\tilde{\vec{J}}}_{\vec{k}\omega}^* \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega} - \left(\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega}^*\right)\left(\hat{n} \cdot \tilde{\tilde{\vec{J}}}_{\vec{k}\omega}\right)\right]$$

E.9. A current $I(t)$ flows in a ring of radius $\underline{a}$ in the xy-plane. The current has time dependence

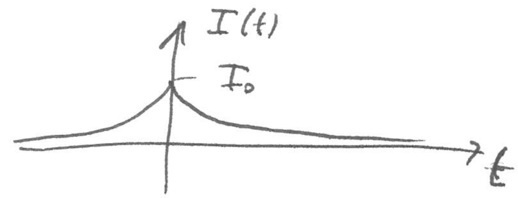
$$I(t) = \frac{I_0}{1+\left(\frac{t}{\tau_0}\right)^2} \qquad \text{where } I_0 \text{ and } \tau_0 \text{ are constants.}$$

In spherical polar coordinates the current density is

$$\vec{J}(\vec{r},t) = I(t)\,\hat{e}_\varphi\,\delta(r-a)\,\delta\left(\theta-\tfrac{\pi}{2}\right)\frac{1}{a}$$



Find an expression for the total energy radiated from the system.



$$\tilde{\vec{J}}_\omega(\vec{r}) = \tilde{I}_\omega\,\hat{e}_\varphi\,\delta(r-a)\,\delta\left(\theta-\tfrac{\pi}{2}\right)\frac{1}{a}$$

$$\text{where } \tilde{I}_\omega = \int_{t=-\infty}^{\infty} I(t)\,e^{i\omega t}\,dt = I_0\int_{t=-\infty}^{\infty}\frac{e^{i\omega t}\,dt}{1+\left(\frac{t}{\tau_0}\right)^2} = 2I_0\tau_0\int_{u=0}^{\infty}\frac{\cos(\omega u\tau_0)\,du}{1+u^2}$$

$$\tilde{I}_\omega = \pi I_0\tau_0\,e^{-|\omega|\tau_0}$$

$$\tilde{\vec{J}}_\omega(\vec{r}) = \pi I_0\tau_0\,e^{-|\omega|\tau_0}\,\hat{e}_\varphi\,\delta(r-a)\,\delta\left(\theta-\tfrac{\pi}{2}\right)\frac{1}{a}$$

Now we need

$$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = \iiint \tilde{\vec{J}}_\omega(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, dV$$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \left[\pi I_0 \tau_0 e^{-|\omega|\tau_0}\, \hat{e}_\varphi\, \delta(r-a)\,\delta(\theta-\tfrac{\pi}{2})\,\tfrac{1}{a}\right] e^{-i\vec{k}\cdot\vec{r}}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Choose the coordinate system so that $\hat{z}$ is normal to the plane of the loop (in the right-hand sense) and choose the $\hat{x}$ direction such that $\vec{k}$ is in the xz-plane.

then $k_z \equiv k\cos\theta_k$, $k_x \equiv k\sin\theta_k$, $k_y \equiv 0$.

$$\vec{k}\cdot\vec{r} = kr\left(\sin\theta_k\, \sin\theta\cos\varphi + \cos\theta_k\, \cos\theta\right)$$

In the spacial integral above, the delta function will select $r=a$ and $\theta=\frac{\pi}{2}$ for which $\cos\theta=0$ and $\sin\theta=1$.

$$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = \pi I_0 \tau_0 e^{-|\omega|\tau_0}\, a \int_{\varphi=0}^{2\pi} e^{-ika\sin\theta_k\cos\varphi}\, \hat{e}_\varphi\, d\varphi$$

$$= \pi I_0 \tau_0 e^{-|\omega|\tau_0}\, a \int_{\varphi=0}^{2\pi} e^{-ika\sin\theta_k\cos\varphi} \left(\hat{y}\cos\varphi - \hat{x}\sin\varphi\right) d\varphi$$

But $\int_{\varphi=0}^{2\pi} e^{-ika\sin\theta_K \cos\varphi} \sin\varphi \, d\varphi = 0$ because $\sin\varphi$ is odd.

Define $u \equiv ka \sin\theta_K$

$$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = \pi I_0 \tau_0 e^{-|\omega|\tau_0} a\hat{y} \int_{\varphi=-\pi}^{+\pi} \cos\varphi \, e^{-iu\cos\varphi} d\varphi$$

$$= \pi I_0 \tau_0 e^{-|\omega|\tau_0} a\hat{y} \, i \frac{d}{du} \int_{\varphi=-\pi}^{+\pi} e^{-iu\cos\varphi} d\varphi$$

Now recognize $J_0(u) = \frac{1}{2\pi} \int_{\varphi=-\pi}^{+\pi} e^{iu\cos\varphi} d\varphi$

is the Bessel function of order zero.

So $$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = 2\pi^2 I_0 \tau_0 e^{-|\omega|\tau_0} a\hat{y} \, i \underbrace{\frac{d}{du} J_0(u)}_{-J_1(u)}$$

$$\tilde{\tilde{\vec{J}}}_{\vec{k}\omega} = -i 2\pi^2 I_0 \tau_0 e^{-|\omega|\tau_0} a\hat{y} \, J_1(ka \sin\theta_K)$$