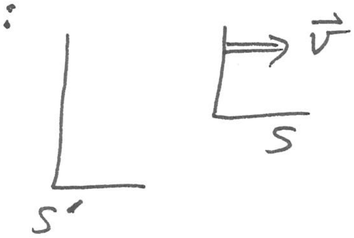
An example of <u>Galilean</u> relativity:

Newton's Laws of Mechanics are invariant under the coordinate transformation:

$$x' = x + vt \quad , \quad t' = t$$

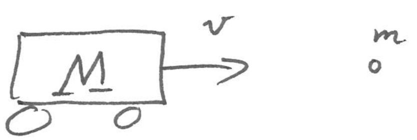


↑ time is absolute
events that are simultaneous in one frame will be simultaneous in <u>all</u> frames.

$$\frac{dx'}{dt} = \frac{dx}{dt} + v$$

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} \qquad F' = m\frac{d^2x'}{dt^2} = m\frac{d^2x}{dt^2} = F$$

Consider the collision of a truck moving at 60 mph hitting a ping-pong ball at rest.



Transform to the frame of the driver. The truck is at rest and the ball approaches at 60 mph

After the collision in this frame, the ball recedes at 60 mph

Transform back to the original frame. The small mass ball acquires <u>twice</u> the speed of the large mass truck.

Although Newton's Laws of mechanics are invariant under Galilean transformations, Maxwell's equations are <u>not</u>.

Possibilities:

① Maxwell's equations are wrong. The correct equations of electrodynamics are invariant under Galilean transformations.

② Galilean relativity is appropriate for mechanics, but electrodynamics is invariant under a different transformation. In particular, there is a preferred frame in which the speed of waves is $\frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$. In this frame, the "luminiferous aether" is at rest.

③ Mechanics and electrodynamics <u>both</u> obey the same relativity principle, but it is not Galilean relativity. Newton's laws need to be modified despite their great success in planetary motion and <u>predicting</u> Neptune.

Special Relativity

(which can deal with acceleration perfectly well, by the way)

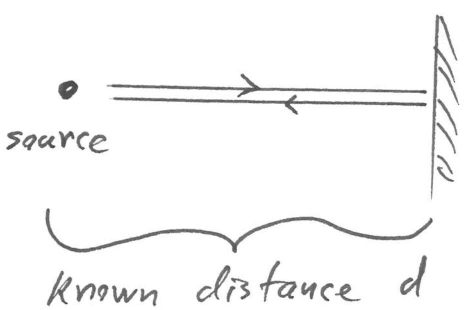
The Special Theory of Relativity (that is, relativity without gravitation) is concerned with the comparison of physical phenomena by observers in different inertial frames. An inertial reference frame or inertial coordinate system is one in which a free particle moves with uniform velocity. Since we are ignoring gravitation, an operational definition of an inertial frame can be taken to be any frame which moves uniformly with respect to the "fixed stars." Clearly there is an infinite number of inertial frames.

Each coordinate frame contains devices that measure distance and time. We assume that space is Euclidean or "flat" (it is <u>not</u> if gravity is included) so that it is possible to introduce Cartesian coordinates with the distance between two points given by

$$\Delta l = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

Coordinate systems in the inertial frame are then constructed using rulers that are calibrated against some standard (such as some chosen wavelength of light in the inertial frame).

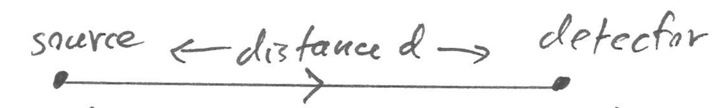
Clocks are cyclic devices that are calibrated against some standard (such as the mean life time of a decaying elementary particle at rest in the inertial frame). We can measure the speed of light in a particular frame of reference by the reflection process



t = total transit time

$$c = \frac{2d}{t}$$

We can <u>synchronize</u> clocks within a particular coordinate frame by the following procedure:

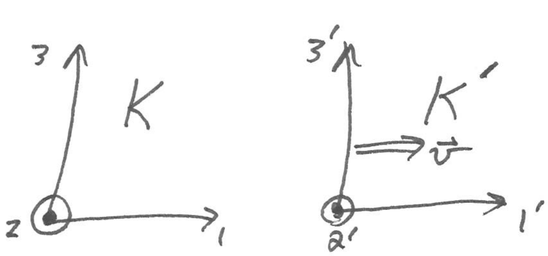


Send signal at $t=0$

set clock to time $t = \frac{d}{c}$ upon receipt of the signal

In this way, all clocks within a coordinate frame are assumed to have been synchronized. We assume this has been done in all inertial frames with which we will deal.

Next we consider how two observers in different inertial frames compare their observations of a particular event. What is the transformation law if we give up the Galilean/Newtonian idea of universal simultaneity? We require that the transformation still be <u>linear</u> since that implies that free particles move with constant velocity in all inertial frames.



frame K' moves with velocity $\vec{v}$ along the x_1-axis of K.

$x_1' = a_1(v)\, x_1 - b(v)\, t$	$x_1 = \bar{a}_1(v)\, x_1' + \bar{b}(v)\, t'$
$x_2' = a_2(v)\, x_2$	$x_2 = \bar{a}_2(v)\, x_2'$
$x_3' = a_3(v)\, x_3$	$x_3 = \bar{a}_3(v)\, x_3'$

Since the origin of K' must move with velocity v along the x_1-axis and the origin of K must move with velocity $-v$ along the x_1'-axis, we must have

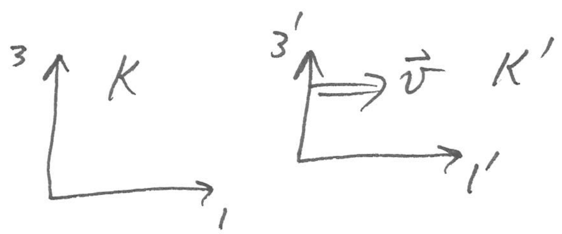
$x_1' = \gamma(v)\,(x_1 - vt)$	$x_1 = \bar{\gamma}(v)\,(x_1' + vt')$
$x_2' = a_2(v)\, x_2$	$x_2 = \bar{a}_2(v)\, x_2'$
$x_3' = a_3(v)\, x_3$	$x_3 = \bar{a}_3(v)\, x_3'$

where we have written $b(v) = v\, a_1(v) = v\, \gamma(v)$

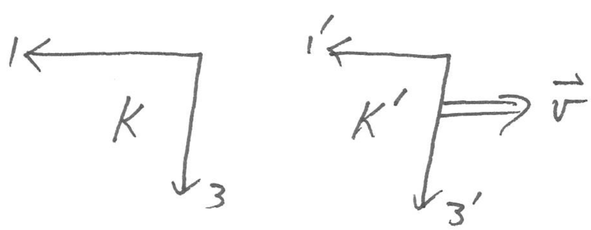
and $\bar{b}(v) = v\, \bar{a}_1(v) \equiv v\, \bar{\gamma}(v)$.

Comparing the directions perpendicular to the motion, we find $\bar{a}_2(v) = \dfrac{1}{a_2(v)}$ and $\bar{a}_3(v) = \dfrac{1}{a_3(v)}$.

The transformation equations describe this picture



Now reverse the x_1, x_3, x_1', x_3' axes



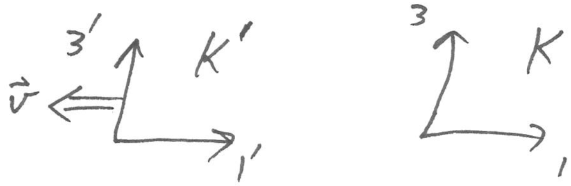
Since $\gamma(v)$, $\bar{\gamma}(v)$, $a_2(v)$ and $a_3(v)$ depend only on the relative velocity between the frames, we have:

$$
\begin{aligned}
x_1' &= \gamma(v)\,(x_1 + vt) \\
x_2' &= a_2(v)\, x_2 \\
x_3' &= a_3(v)\, x_3
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
x_1 &= \bar{\gamma}(v)\,(x_1' + vt') \\
x_2 &= \frac{1}{a_2(v)}\, x_2' \\
x_3 &= \frac{1}{a_3(v)}\, x_3'
\end{aligned}
\qquad (*)
$$

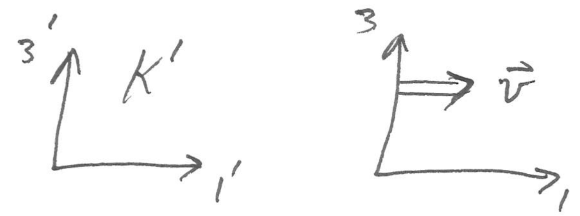
We next invoke the principle of isotropy of space (we really did this already when we assumed that the speed of light was measured in a given coordinate frame by reflection.

That is, the speed of light does not depend on the direction in space.

Thus we can rotate the picture above to obtain



which is equivalent to



But in addition to being described by the last set of transformation equations, this picture must also be described by

$$x_1 = \gamma(v)\,(x_1' - v t') \qquad\qquad x_1' = \bar{\gamma}(v)\,(x_1 + v t)$$

$$x_2 = a_2(v)\, x_2' \qquad\qquad x_2' = \frac{1}{a_2(v)}\, x_2 \qquad\qquad (**)$$

$$x_3 = a_3(v)\, x_3' \qquad\qquad x_3' = \frac{1}{a_3(v)}\, x_3$$

Comparing the equations (*) and (**) we find

$$\frac{1}{a_2(v)} = a_2(v) \quad , \quad \frac{1}{a_3(v)} = a_3(v) \quad , \quad \bar{\gamma}(v) = \gamma(v)$$

Thus $a_2^2(v) = 1 \implies a_2(v) = \pm 1$ and so for $a_3(v)$.

Since at zero relative velocity $a_2(0) = 1$, it must be that $a_2(v) = +1$ and $a_3(v) = +1$.

So far we have

$$x_1' = \gamma(v)(x_1 - vt) \qquad\qquad x_1 = \gamma(v)(x_1' + vt')$$

$$x_2' = x_2$$

$$x_3' = x_3$$

Solving this last equation for t' we obtain

$$t' = \frac{1}{v}\left[\frac{x_1}{\gamma(v)} - x_1'\right] = \frac{1}{v}\left[\frac{x_1}{\gamma(v)} - \gamma(v)(x_1 - vt)\right]$$

$$= \gamma(v)\, t - \frac{1}{v}\left(\gamma(v) - \frac{1}{\gamma(v)}\right) x_1$$

If we define a quantity $V^2(v)$ through

$$V^2(v) \equiv \frac{v^2}{1 - \frac{1}{\gamma^2(v)}} \qquad \text{or} \qquad \gamma^2(v) = \frac{1}{1 - \frac{v^2}{V^2(v)}}$$

then we have

$$t' = \gamma(v)\left[t - \frac{v}{V^2(v)}\, x_1\right]$$

Note that we must demand $V(v) > v$ otherwise we would have $\gamma(v)$ imaginary which wouldn't make sense. But what is $V(v)$?

Suppose we have three inertial frames all with parallel axes and moving along the various 1-axes. We suppress the 1 index, ignore the trivial 2- and 3-directions, and use the labels (a, b, c) instead of unprimed, single prime ', and double prime ".

K_b moves relative to K_a
K_c moves relative to K_b

K_c also moves relative to K_a

$$x_b = \gamma_{ba}\left(x_a - v_{ba} t_a\right) \qquad t_b = \gamma_{ba}\left(t_a - \frac{v_{ba}}{V_{ba}^2} x_a\right)$$

$$x_c = \gamma_{cb}\left(x_b - v_{cb} t_b\right) \qquad t_c = \gamma_{cb}\left(t_b - \frac{v_{cb}}{V_{cb}^2} x_b\right)$$

$$x_c = \gamma_{ca}\left(x_a - v_{ca} t_a\right) \qquad t_c = \gamma_{ca}\left(t_a - \frac{v_{ca}}{V_{ca}^2} x_a\right)$$

where $\gamma_{ba} = \dfrac{1}{\sqrt{1 - \dfrac{v_{ba}^2}{V_{ba}^2}}}$ and so for γ_{cb} and γ_{ca}.

Clearly, there is a question of consistency that must be met in these three transformations. This is the group property that two successive transformations yields yet another transformation

$$x_c = \gamma_{cb}(x_b - v_{cb} t_b)$$ substitute for x_b and t_b

$$x_c = \gamma_{cb}\left[\gamma_{ba}(x_a - v_{ba} t_a) - v_{cb}\gamma_{ba}\left(t_a - \frac{v_{ba}}{V_{ba}^2} x_a\right)\right]$$

$$t_c = \gamma_{cb}\left(t_b - \frac{v_{cb}}{V_{cb}^2} x_b\right)$$ substitute for x_b and t_b

$$t_c = \gamma_{cb}\left[\gamma_{ba}\left(t_a - \frac{v_{ba}}{V_{ba}^2} x_a\right) - \frac{v_{cb}}{V_{cb}^2}\gamma_{ba}(x_a - v_{ba} t_a)\right]$$

So

$$x_c = \gamma_{cb}\gamma_{ba}\left(1 + \frac{v_{cb} v_{ba}}{V_{ba}^2}\right)\left[x_a - \frac{v_{ba} + v_{cb}}{1 + \frac{v_{cb} v_{ba}}{V_{ba}^2}} t_a\right]$$

and

$$t_c = \gamma_{cb}\gamma_{ba}\left(1 + \frac{v_{cb} v_{ba}}{V_{cb}^2}\right)\left[t_a - \frac{\frac{v_{ba}}{V_{ba}^2} + \frac{v_{cb}}{V_{cb}^2}}{1 + \frac{v_{cb} v_{ba}}{V_{cb}^2}} x_a\right]$$

But we must also have

$$x_c = \gamma_{ca}(x_a - v_{ca} t_a)$$ and $$t_c = \gamma_{ca}\left(t_a - \frac{v_{ca}}{V_{ca}^2} x_a\right)$$

So we can match coefficients

set $t_a = 0$ and look at the coefficients of x_a in both equations for x_c:

$$\gamma_{ca} = \gamma_{cb}\gamma_{ba}\left(1 + \frac{v_{cb}v_{ba}}{V^2_{ba}}\right)$$

Now set $x_a = 0$ and look at the coefficients of t_a in both equations for t_c:

$$\gamma_{ca} = \gamma_{cb}\gamma_{ba}\left(1 + \frac{v_{cb}v_{ba}}{V^2_{cb}}\right)$$

To be consistent $V^2_{ba} = V^2_{cb} \equiv V^2$ (defines V^2)

It also follows from the coefficient of t_a in the equation for x_c:

$$v_{ca} = \frac{v_{ba} + v_{cb}}{1 + \frac{v_{cb}v_{ba}}{V^2}}$$

(new velocity addition law. Galilean is $v_{ca} = v_{ba} + v_{cb}$)

and from the coefficient of x_a in the equation for t_c:

$$\frac{v_{ca}}{V^2_{ca}} = \frac{1}{V^2}\frac{v_{ba} + v_{cb}}{1 + \frac{v_{cb}v_{ba}}{V^2}} = \frac{1}{V^2}v_{ca} \Longrightarrow V^2_{ca} = V^2$$

In summary:

$$V^2_{ba} = V^2_{cb} = V^2_{ca} \equiv V^2$$ independent of frame, a <u>universal</u> constant.

We can also see how the γ factors combine:

$$\gamma_{ca} = \gamma_{cb}\,\gamma_{ba}\left(1 + \frac{v_{cb}\,v_{ba}}{V^2}\right)$$

We will need to perform an experiment to see what this largest speed is. Remember $V > v_{ba}$ for <u>any</u> relative frame velocity v_{ba}.

Our new non-Galilean transformations so far are:

$$x' = \gamma(v)(x - vt) \qquad\qquad x = \gamma(v)(x' + vt')$$

$$t' = \gamma(v)\left(t - \frac{v}{V^2}x\right) \qquad\qquad t = \gamma(v)\left(t' + \frac{v}{V^2}x'\right)$$

This is called the <u>Lorentz</u> <u>transformation</u>,

$$\text{with } \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{V^2}}} \qquad (1 \le \gamma < \infty)$$

By the way, if the universal constant V, the ultimate speed, is ∞, then $\gamma \to 1$ and $t = t'$ and we recover the Galilean transformation.

Our derived transformation is the <u>most</u> <u>general</u> linear transformation allowed and V could be finite.

Consequences of the Lorentz transformation

① Put a clock at rest in K' on the x_1'-axis.

Then $\Delta x' = 0$.

$$\Delta t = \gamma(v)\,\Delta t'$$

Define $\Delta t' \equiv \tau_0$ the proper time, the time read by a clock at rest in an inertial frame. Call τ the elapsed time in K as the clock moves down the x_1-axis. Note that this requires the recording of time on a sequence of different clocks in K (all of which were synchronized in K). The clocks in K will not be synchronized according to an observer in K'.

We find

$$\tau = \gamma(v)\,\tau_0 = \frac{\tau_0}{\sqrt{1 - \frac{v^2}{V^2}}} \geq \tau_0$$

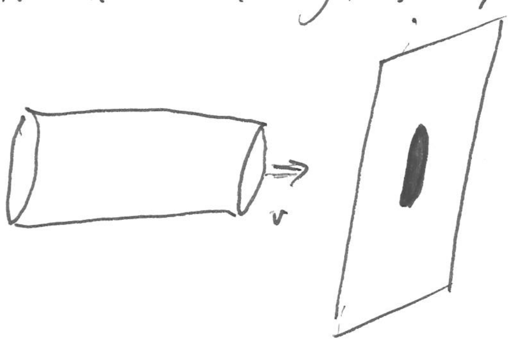
This is time dilation (if $v > 0$). Moving clocks run slow. If you and I move relative to one another, I see your clock running slow while you see my clock running slow, but this is not a paradox.

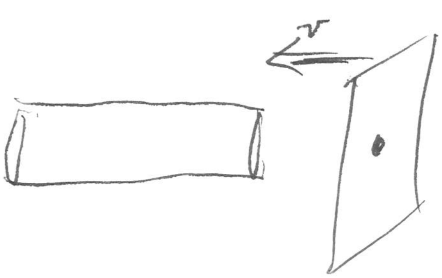
(2) Put a rod at rest in K' and simultaneously locate both of its ends for a length measurement in K. Then $\Delta t = 0$ and $\Delta x' = \gamma(v)\, \Delta x$. Define the length of the rod in K' as L_0, the proper length; and the length of the rod in K is L. Then

$$L = \frac{L_0}{\gamma(v)} = L_0 \sqrt{1 - \frac{v^2}{V^2}} \le L_0$$

This is length contraction. Moving objects are shrunk in the direction of $\vec{v}$. If we have relative motion, I will see your ship contracted in the direction of motion while you will see my ship contracted. Again, this is not a paradox. (Ladder and barn)

How about lengths perpendicular to the velocity?





If lengths shrank in the y- and z-directions, someone on the barrier would predict that the cylinder would pass easily through the hole, but someone on the cylinder would predict a crash.