It is tempting to say that all the quantities defined above — A^μ, J^μ, and $F^{\mu\nu}$ are 4-tensors, but that is a matter of physics, not merely definition. We now look at the quantities in question are show that they really are tensors under Lorentz transformations.

(i) J^μ - 4 current density

Consider a small amount of charge at rest δq. We assume a continuous charge distribution. Then the charge density is

$$\rho_0 = \frac{\delta q}{\delta V_0} \qquad \text{(the o subscript means "proper")}$$

where δV_0 is the volume element in which δq sits. If we now move the volume element with velocity $\vec{u}$, it will contract in the direction of motion with

$$\delta V = \sqrt{1 - \frac{u^2}{c^2}}\, \delta V_0 \qquad \text{and charge density}$$

$$\rho = \frac{\delta q}{\delta V} = \gamma \rho_0$$

Note that we have taken δq as Lorentz invariant. Conservation of charge actually requires this. The current density is

$$\vec{J} = \rho \vec{u} = \gamma \rho_0 \vec{u}$$

Thus if we define $J^0 = c\rho$ then

$$J^\mu = \rho_0 \mathbb{U}^\mu$$ where $\mathbb{U}^\mu$ is the local velocity field of the charged "fluid".

If we have more than one species of charged fluid then

$$J^\mu_{(x)} = \sum_s \rho_0^{(s)}{}_{(x)} \, \mathbb{U}^\mu_{(s)}{}_{(x)}$$

Since $\rho_0^{(s)}{}_{(x)}$ is a scalar field — it is the charge density of the s'th species in a frame which is instantaneously at rest with respect to the local charge of species s — and $\mathbb{U}^\mu_{(s)}(x) = \left[\gamma_{(s)}(x)\, c,\ \gamma_{(s)}(x)\, \vec{u}_{(s)}(x)\right]$ is a 4-vector velocity field, then $J^\mu_{(x)}$ is also a 4-vector field.

(ii) $F^{\mu\nu}$

Quite aside from Maxwell's equations, the electric and magnetic fields are defined operationally through the electromagnetic force. The force law is

$$\frac{d\vec{\mathbb{P}}}{dt} = q\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right) \qquad \text{(c.g.s. units)} \qquad (*)$$

Since the magnetic force is perpendicular to the particle velocity, then only the electric force does work.

Thus $\dfrac{dK}{dt} = q\,\vec{u}\cdot\vec{E}$ where K is the kinetic energy of the particle. Since the rest mass of the particle is a constant, we may write

$$\frac{d\mathbb{P}^0}{dt} = \frac{q}{c}\,\vec{u}\cdot\vec{E} \qquad (**)$$

The equations (*) and (**) can be combined into

$$\frac{d\mathbb{P}^\mu}{dt} = \frac{q}{c}\,\frac{dx_\nu}{dt}\,F^{\mu\nu}$$

or multiplying both sides by $\gamma = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ we have

$$\frac{d\mathbb{P}^\mu}{d\tau} = \frac{q}{c}\,\mathbb{U}_\nu F^{\mu\nu}$$

Now the left-hand side is known to be a 4-vector and $\mathbb{U}_\nu$ is an arbitrary 4-vector. It therefore follows from the following theorem that $F^{\mu\nu}$ is a second rank 4-tensor.

<u>Homework</u>: Prove that if A_μ is an arbitrary covariant 4-vector and B^μ is a contravariant 4-vector and $C^{\mu\nu}A_\nu = B^\mu$, then $C^{\mu\nu}$ is a contravariant 4-tensor

Thus Maxwell's equations

$$\partial_\mu F^{\mu\nu}_{(x)} = \frac{4\pi}{c} J^\nu_{(x)}$$

$$\partial^\mu F^{\alpha\beta}_{(x)} + \partial^\alpha F^{\beta\mu}_{(x)} + \partial^\beta F^{\mu\alpha}_{(x)} = 0$$

hold in all inertial frames. Remember ∂_μ and ∂^μ just convert tensors into higher rank tensors and index contraction reduces the tensor rank by 2.

Field invariants

From the space time 4-vector, we constructed the invariant interval between events

$$\Delta s^2 = \Delta x^\mu \Delta x_\mu$$

Δs^2 has the same value in all frames. There are also invariants for the electromagnetic field. We must construct scalars from $F^{\mu\nu}$.

(i) linear invariants

$F_{(x)}{}^{\mu}{}_{\mu}$ is a scalar field, but

$$F^\mu{}_\mu = F^0{}_0 + \sum_{k=1}^{3} F^k{}_k = F^{00} - \sum_k F^{kk}$$

however $F^{\mu\mu}$ (not summed) $= 0$ since $F^{\mu\nu}$ is antisymmetric. Therefore $F^\mu{}_\mu = 0$.

(ii) Quadratic invariants

$F_{(x)}^{\mu\nu} F_{(x)\mu\nu}$ is a scalar field. We have

$$F_{\mu\nu} = g_{\mu\alpha}\, g_{\nu\beta}\, F^{\alpha\beta} \quad \text{with}$$

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \quad \text{as a matrix}$$

From this matrix and the one given previously for $[F^{\mu\nu}]$ we calculate

$$-\tfrac{1}{2} F^{\mu\nu} F_{\mu\nu} = +\tfrac{1}{2} F^{\mu\nu} F_{\nu\mu} = \tfrac{1}{2} \operatorname{Tr}\left([F^{\mu\nu}] \cdot [F_{\nu\mu}]\right)$$

$$= \vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}^{\,2} = E^2 - B^2$$

Thus $E^2 - B^2$ is a Lorentz invariant. If $E > B$ in one inertial frame, then $E > B$ in <u>all</u> inertial frames.

Now consider the totally antisymmetric quantity

$$\epsilon_{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } \mu\nu\alpha\beta \text{ even permutation of } 0123 \\ -1 & \text{if } \mu\nu\alpha\beta \text{ odd permutation of } 0123 \\ 0 & \text{if any of } \mu\nu\alpha\beta \text{ are the same} \end{cases}$$

We require this symbol to have the same definition in all inertial frames. Thus $\epsilon_{\mu\nu\alpha\beta}$ must transform according to the rule

$$\bar{\epsilon}_{\mu\nu\alpha\beta} = \det[\underline{\underline{L}}]\, l_\mu{}^\sigma\, l_\nu{}^\tau\, l_\alpha{}^\gamma\, l_\beta{}^\rho\, \epsilon_{\sigma\tau\gamma\rho}$$

where $[\underline{\underline{L}}]$ is the matrix of coefficients $l_\mu{}^\nu$.

But the permutation symbol can be used to <u>define</u> the determinant of a matrix through the relation

$$l_\mu{}^\sigma\, l_\nu{}^\tau\, l_\alpha{}^\gamma\, l_\beta{}^\rho\, \epsilon_{\sigma\tau\gamma\rho} = \det[\underline{\underline{L}}]\, \epsilon_{\mu\nu\alpha\beta}$$

Thus we have

$$\bar{\epsilon}_{\mu\nu\alpha\beta} = \left(\det[\underline{\underline{L}}]\right)^2 \epsilon_{\mu\nu\alpha\beta}$$

But $l^\mu{}_\alpha\, l_\mu{}^\beta = \delta_\alpha^\beta$

$$\left(g^{\mu\sigma}\, l_\sigma{}^\rho\, g_{\rho\alpha}\right) l_\mu{}^\beta = \delta_\alpha^\beta$$

$$g_{\alpha\rho}\, (l^T)^\rho{}_\sigma\, g^{\sigma\mu}\, l_\mu{}^\beta = \delta_\alpha^\beta$$

adjacent indices summed over $\Rightarrow$ matrix multiplication

$$\underline{\underline{G}}\, \underline{\underline{L}}^T\, \underline{\underline{G}}\, \underline{\underline{L}} = \underline{\underline{I}}$$

where $[g_{\alpha\beta}] = [g^{\sigma\mu}] = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

and $\underline{\underline{L}} = \begin{pmatrix} l_0{}^0 & l_0{}^1 & l_0{}^2 & l_0{}^3 \\ l_1{}^0 & l_1{}^1 & l_1{}^2 & l_1{}^3 \\ l_2{}^0 & l_2{}^1 & l_2{}^2 & l_2{}^3 \\ l_3{}^0 & l_3{}^1 & l_3{}^2 & l_3{}^3 \end{pmatrix}$

$$\underline{\underline{G}}\,\underline{\underline{L}}^T\,\underline{\underline{G}}\,\underline{\underline{L}} = \underline{\underline{I}}$$

$$\underbrace{[\det(\underline{\underline{G}})]^2}_{1}\ \underbrace{\det(\underline{\underline{L}}^T)}_{=\det(\underline{\underline{L}})}\det(\underline{\underline{L}}) = 1$$

$$\Rightarrow [\det(\underline{\underline{L}})]^2 = 1 \quad \Rightarrow \quad \det(\underline{\underline{L}}) = \pm 1$$

The equation was

$$\bar{\epsilon}_{\mu\nu\alpha\beta} = [\det(\underline{\underline{L}})]^2\,\epsilon_{\mu\nu\alpha\beta}$$

So $\bar{\epsilon}_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta}$ frame independent as required.

A quantity which transforms according to

$$T^{\bar{\mu}_1 \cdots \bar{\mu}_m}{}_{\bar{\nu}_1 \cdots \bar{\nu}_n} = \det(\underline{\underline{L}})\, l^{\bar{\mu}_1}{}_{\mu_1} \cdots l^{\bar{\mu}_m}{}_{\mu_m}\, l_{\bar{\nu}_1}{}^{\nu_1} \cdots l_{\bar{\nu}_n}{}^{\nu_n}\, T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n}$$

is called a <u>pseudo tensor</u> of rank $m+n$. It differs from a tensor by the factor $\det(\underline{\underline{L}})$. Note if $\det(\underline{\underline{L}}) = +1$ then there is no difference. $\det(\underline{\underline{L}}) = -1$ arises from space inversion or time reversal (but both space inversion and time reversal together give $\det(\underline{\underline{L}}) = +1$. Some times the word "orthochronous" is used to indicate no time reversal).

The quantity $F^{\mu\nu}$ is a tensor. The quantity

$$\mathcal{F}_{\mu\nu} \equiv \frac{1}{2}\, \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$$

is a second rank pseudo tensor which is said to be "dual" to $F^{\mu\nu}$.

<u>Aside</u>: In ~~three~~ space dimensions under $SO(3)$ rotations

ϵ_{ijk} (Levi-Civita symbol) is a 3rd rank pseudo tensor and $A_i = \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_{jk}$ is a pseudovector which is dual to the antisymmetric second rank tensor A_{jk}

$$\underline{\underline{A}} = \begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}$$

$$[\mathcal{F}_{\mu\nu}] = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & -E^3 & E^2 \\ B^2 & E^3 & 0 & -E^1 \\ B^3 & -E^2 & E_1 & 0 \end{pmatrix}$$

and we can construct the contravariant form

$$\mathcal{F}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \mathcal{F}_{\alpha\beta} \quad , \quad [\mathcal{F}^{\mu\nu}] = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & -E^3 & E^2 \\ -B^2 & E^3 & 0 & -E^1 \\ -B^3 & -E^2 & E^1 & 0 \end{pmatrix}$$

Next we form the invariant with regard to proper (no space inversion or time reversal) Lorentz transformation

$$\frac{1}{4} \mathcal{F}_{\mu\nu} F^{\mu\nu} = \vec{E} \cdot \vec{B}$$

Thus $\vec{E} \cdot \vec{B}$ is invariant under proper Lorentz transformations. If $\vec{E} \cdot \vec{B} = 0$ in one frame, then $\vec{E} \cdot \vec{B} = 0$ in all inertial frames. In particular, suppose $\vec{B} = 0$ in some frame and $\vec{E} \neq 0$. Then in all other frames $\vec{E}' \cdot \vec{B}' = 0$, that is $\vec{E}'$ and $\vec{B}'$ are perpendicular. And we can't find a frame in which $\vec{E}' = 0$ since $E^2 - B^2 > 0$ in the original frame and $E^2 - B^2$ is a Lorentz invariant, thus $E'^2 > B'^2$ in the new frame.

Under improper Lorentz transformations (spatial inversion or time reversal) then $\vec{E}\cdot\vec{B}$ changes sign.

Under inversion, the vector $\vec{E}$ does <u>not</u> change sign (the components E^i do, but so do the basis vectors).

However, from $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ we see that $\vec{B}$ is a pseudovector ("axial" vector) – that is, the components B^k do not change sign under inversion, but the basis vectors do, so $\vec{B}$ changes sign under inversion. Similarly under time reversal, $\vec{J}$ changes sign and hence $\vec{B}$ again changes sign.

Since $F^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$ are determined in terms of $\vec{E}$ and $\vec{B}$, then all Lorentz scalars (or pseudoscalars) must be made out of $\vec{E}\cdot\vec{B}$ or E^2-B^2.

Note that E^2+B^2 is an energy density and is the 00 component of a second rank tensor – the energy-momentum stress tensor $T^{\mu\nu}$. thus

$$\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \propto E^2-B^2$$

$$\mathcal{F}^{\mu\nu}F_{\mu\alpha}F^{\alpha}{}_{\nu} = 0 \qquad \text{etc.}$$

There are no other invariants besides $\vec{E}\cdot\vec{B}$ and E^2-B^2 (well... and 0).

Behavior of Fields under Lorentz Transformation

Because our experience and intuition about the electromagnetic field is with $\vec{E}$ and $\vec{B}$, and not $F^{\mu\nu}$, we give the transformation law for the fields directly. It is simplest to use a matrix notation. For the special Lorentz transformation (parallel axes with origin of $\bar{K}$ moving with velocity $\vec{v}$ along the 1-axis of frame K), we have

$$\underline{\underline{L}} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \bar{\vec{\mathbb{X}}} = \underline{\underline{L}}\, \vec{\mathbb{X}}$$

$$\vec{\mathbb{X}} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Let $\underline{\underline{F}}$ be the matrix $[F^{\mu\nu}]$ with contravariant components

$$\underline{\underline{\bar{F}}} = \underline{\underline{L}}\, \underline{\underline{F}}\, \underline{\underline{L}}^{T} \longleftrightarrow \bar{F}^{\mu\nu} = l^{\mu}{}_{\alpha}\, l^{\nu}{}_{\beta}\, F^{\alpha\beta}$$

$$= l^{\mu}{}_{\alpha}\, F^{\alpha\beta} (l^{T})_{\beta}{}^{\nu}$$

$$\begin{pmatrix} 0 & -\bar{E}^1 & -\bar{E}^2 & -\bar{E}^3 \\ \bar{E}^1 & 0 & -\bar{B}^3 & \bar{B}^2 \\ \bar{E}^2 & \bar{B}^3 & 0 & -\bar{B}^1 \\ \bar{E}^3 & -\bar{B}^2 & \bar{B}^1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{\bar{F}} = \begin{pmatrix} 0 & -E^1 & -\gamma[E^2-\beta B^3] & -\gamma[E^3+\beta B^2] \\ E^1 & 0 & -\gamma[B^3-\beta E^2] & \gamma[B^2+\beta E^3] \\ \gamma[E^2-\beta B^3] & \gamma[B^3-\beta E^2] & 0 & -B^1 \\ \gamma[E^3+\beta B^2] & -\gamma[B^2+\beta E^3] & B^1 & 0 \end{pmatrix}$$

Thus we have along the direction of motion:

$$\bar{E}^1 = E^1 \quad \text{and} \quad \bar{B}^1 = B^1$$

and perpendicular to the direction of motion:

$$\bar{E}^2 = \gamma[E^2 - \beta B^3] \qquad \bar{B}^2 = \gamma[B^2 + \beta E^3]$$

$$\bar{E}^3 = \gamma[E^3 + \beta B^2] \qquad \bar{B}^3 = \gamma[B^3 - \beta E^2]$$

If you want the inverse transformation, switch barred and unbarred fields and change the sign of $\beta = \frac{v}{c}$.

If the boost direction is not aligned along the 1-axis:

$$\vec{\bar{E}}_{\parallel} = \vec{E}_{\parallel} \quad \text{and} \quad \vec{\bar{B}}_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{\bar{E}}_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta}\times\vec{B}) \qquad \vec{\bar{B}}_{\perp} = \gamma(\vec{B}_{\perp} - \vec{\beta}\times\vec{E})$$

where $\vec{E}_{\parallel} = \dfrac{(\vec{E}\cdot\vec{v})\vec{v}}{v^2}$ and $\vec{E}_{\perp} = \vec{E} - \vec{E}_{\parallel} = \dfrac{\vec{v}\times(\vec{E}\times\vec{v})}{v^2}$

$$\vec{\bar{E}} = (1-\gamma)\,\frac{\vec{v}(\vec{v}\cdot\vec{E})}{v^2} + \gamma\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$$\vec{\bar{B}} = (1-\gamma)\,\frac{\vec{v}(\vec{v}\cdot\vec{B})}{v^2} + \gamma\left(\vec{B} - \frac{\vec{v}}{c}\times\vec{E}\right)$$

$$\vec{E} = (1-\gamma)\,\frac{\vec{v}(\vec{v}\cdot\vec{\bar{E}})}{v^2} + \gamma\left(\vec{\bar{E}} - \frac{\vec{v}}{c}\times\vec{\bar{B}}\right)$$

$$\vec{B} = (1-\gamma)\,\frac{\vec{v}(\vec{v}\cdot\vec{\bar{B}})}{v^2} + \gamma\left(\vec{\bar{B}} + \frac{\vec{v}}{c}\times\vec{\bar{E}}\right)$$

As an example, consider the case of a point charge at rest in frame $\bar{K}$. Then for $v \ll c$, $\gamma \approx 1$ and we have

$$\vec{E} \simeq \vec{\bar{E}} = \frac{kq\,\vec{\bar{r}}}{\bar{r}^3} = kq\,\frac{\vec{r}-\vec{v}t}{|\vec{r}-\vec{v}t|^3} \qquad (k=1 \text{ in c.g.s.})$$

$$\vec{B} \simeq \frac{\vec{v}}{c}\times\vec{\bar{E}} = q\,\frac{\vec{v}}{c}\times\frac{(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3} \qquad (\text{Biot-Savart Law})$$

For a general transformation between inertial frames with <u>low</u> relative velocity we have

$$\vec{\bar{E}} = \vec{E} + \frac{\vec{v}}{c}\times\vec{B} \qquad\Big|\qquad \vec{E} = \vec{\bar{E}} - \frac{\vec{v}}{c}\times\vec{\bar{B}}$$

$$\vec{\bar{B}} = \vec{B} - \frac{\vec{v}}{c}\times\vec{E} \qquad\Big|\qquad \vec{B} = \vec{\bar{B}} + \frac{\vec{v}}{c}\times\vec{\bar{E}}$$

The Galilean invariance of Newton's Second Law gives the transformation above for $\vec{E}$ but <u>not</u> for $\vec{B}$. The reason that Galilean invariance cannot give the correct low velocity transformation law for $\vec{B}$ is that it would require an $\mathcal{O}(\frac{1}{c^2})$ term in Newton's Law and that cannot arise in Newtonian Mechanics. [$\mathcal{O}(\frac{1}{c^2})$ terms are always relativistic corrections whereas $\mathcal{O}(\frac{1}{c})$ are not.]

As an application of the field transformation law for arbitrary velocity between inertial frames we again consider a point charge at rest in $\bar{K}$. Then

$$\vec{\bar{E}} = \frac{q\vec{\bar{r}}}{\bar{r}^3} \quad , \quad \vec{\bar{B}} = 0$$

In frame K, the particle moves with velocity $\vec{v}$.

Suppose motion is in the 1-direction: $v^1 = v$

$$\begin{aligned} \bar{x}^1 &= \gamma(x^1 - vt) \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= x^3 \\ \bar{t} &= \gamma\left(t - \tfrac{v}{c^2}x^1\right) \end{aligned} \qquad \Bigg| \qquad \begin{aligned} x^1 &= \gamma(\bar{x}^1 + v\bar{t}) \\ x^2 &= \bar{x}^2 \\ x^3 &= \bar{x}^3 \\ t &= \gamma\left(\bar{t} + \tfrac{v}{c^2}\bar{x}^1\right) \end{aligned}$$

Suppose you sit in K at $x^1 = 0$, $x^2 = b$, $x^3 = 0$.

In $\bar{K}$, the observer has coordinates

$$\bar{x}^1 = -v\bar{t}, \quad \bar{x}^2 = b, \quad \bar{x}^3 = 0, \quad \bar{t} = \gamma t.$$

hence $\bar{x}^1 = -v\gamma t$ so that

$$\bar{E}^1 = \frac{q\bar{x}^1}{\bar{r}^3}, \quad \bar{E}^2 = \frac{qb}{\bar{r}^3}, \quad \bar{E}^3 = 0, \quad \vec{\bar{B}} = 0.$$

or

$$\bar{E}^1 = -\frac{q\gamma v t}{\bar{r}^3} \quad \text{and} \quad \bar{r} = \sqrt{(\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2}$$

$$\bar{r} = \sqrt{\gamma^2 v^2 t^2 + b^2}$$

hence

$$\bar{E}^1 = \frac{-q\gamma v t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad \bar{E}^2 = \frac{qb}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad \bar{E}^3 = 0,$$

$$\bar{B}^1 = 0, \quad \bar{B}^2 = 0, \quad \bar{B}^3 = 0.$$

The electric field can then be transformed to the K frame:

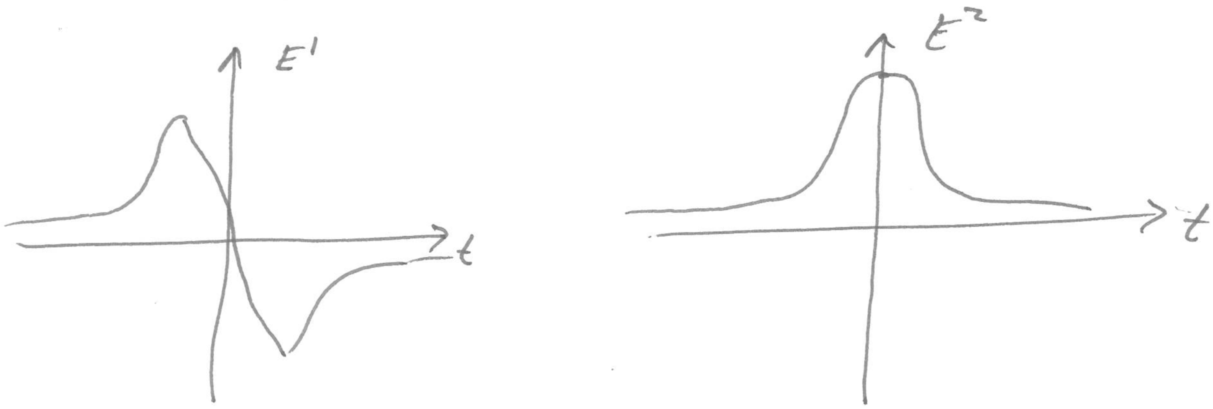
$$E^1 = \bar{E}^1 = \frac{-q\gamma v t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad E^2 = \gamma \bar{E}^2 = \frac{qb\gamma}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad E^3 = 0.$$

For the magnetic field, we have $\vec{\bar{B}} = 0$ so

$$\vec{E} = (1-\gamma)\frac{\vec{v}(\vec{v}\cdot\vec{\bar{E}})}{v^2} + \gamma\vec{\bar{E}} \quad \text{and} \quad \vec{B} = \gamma\frac{\vec{v}}{c}\times\vec{\bar{E}}$$

$$\Rightarrow \vec{B} = \frac{\vec{v}}{c}\times\vec{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v/c & 0 & 0 \\ E^1 & E^2 & 0 \end{vmatrix} \Rightarrow \begin{array}{l} B^1 = 0, \quad B^2 = 0 \\ B^3 = \frac{v}{c}E^2 = \dfrac{qb\gamma\beta}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \end{array}$$

We plot the field components E^1 and E^2 as a function of time. It is straightforward to calculate that $|E^1|$ reaches a maximum value $\frac{2}{\sqrt{27}}\frac{q}{b^2}$ at $t = \frac{\pm b}{\sqrt{2}\, v \gamma}$ while E^2 reaches a maximum value of $\frac{\gamma q}{b^2}$ at $t = 0$:

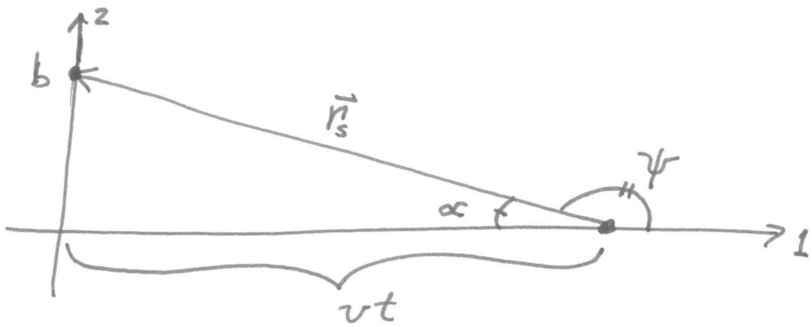


In both cases, the width of the peaks is $\Delta t \sim \frac{b}{\gamma v}$. Because of its inertia, a charged particle detector at $x^1 = 0$, $x^2 = b$, $x^3 = 0$ may well not detect E^1 for $\gamma \gg 1$ ($v \lesssim c$), but it will certainly see E^2.

Next we introduce a vector $\vec{r}_s$ with coordinates

$$x_s^1 = -vt \quad , \quad x_s^2 = b \quad , \quad x_s^3 = 0 .$$

This is a vector drawn from the location of the charge in K at time t to the observation point $(0, b, 0)$.



$$x_s^2 = r_s \sin\alpha = b$$

$$x_s^1 = r_s \cos\alpha = vt$$

From the previous expressions we have

$$E^1 = \frac{q\gamma x_s^1}{\left[\gamma^2 (x_s^1)^2 + (x_s^2)^2\right]^{3/2}} \qquad E^2 = \frac{q\gamma x_s^2}{\left[\gamma^2 (x_s^1)^2 + (x_s^2)^2\right]^{3/2}}$$

$$\Rightarrow \frac{E^1}{E^2} = \frac{x_s^1}{x_s^2} \Rightarrow \vec{E} \text{ is } \underline{\text{radial}} \text{ in } K!$$

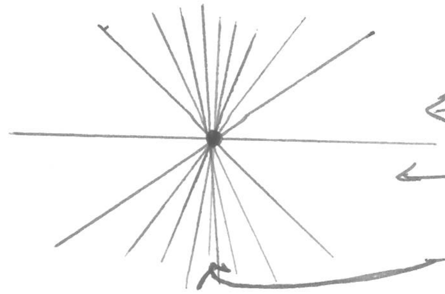
$$\left[\gamma^2 (x_s^1)^2 + (x_s^2)^2\right] = r_s^2\left[\gamma^2 \cos^2\alpha + \sin^2\alpha\right] = r_s^2\left[\gamma^2(1-\sin^2\alpha) + \sin^2\alpha\right]$$

$$= r_s^2\gamma^2\left[1 + \left(\frac{1}{\gamma^2} - 1\right)\sin^2\alpha\right] = r_s^2\gamma^2\left[1 - \frac{v^2}{c^2}\sin^2\alpha\right]$$

$$= r_s^2\gamma^2\left[1 - \frac{v^2}{c^2}\sin^2\psi\right]$$

$$\vec{E} = \frac{q\vec{r}_s}{r_s^3\gamma^2\left(1 - \frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$

Thus the electric field lines in K are radial but <u>anisotropic</u>.



field line spacing opens up because of motion.

field spacing at $v=0$.

Motion in a Uniform Magnetic Field

In the absence of an electric field we have

$$\frac{d\vec{P}}{dt} = q\frac{\vec{u}}{c}\times\vec{B} \quad , \quad \vec{E}=0$$

from which it follows that $\vec{u}\cdot\frac{d\vec{P}}{dt}=0$, but $\vec{P}$ is of the form $\vec{P}=f(u)\,\vec{u}$ where $f(u)=m\gamma=\frac{m}{\sqrt{1-\frac{u^2}{c^2}}}$

Thus $$\frac{d\vec{P}}{dt} = f(u)\frac{d\vec{u}}{dt} + f'(u)\frac{du}{dt}\vec{u}$$

and so

$$\vec{u}\cdot\frac{d\vec{P}}{dt} = \vec{u}\cdot\frac{d\vec{u}}{dt}f(u) + f'(u)\frac{du}{dt}u^2 = 0$$

Now use $\frac{1}{2}d(u^2)=\frac{1}{2}d(\vec{u}\cdot\vec{u}) = \vec{u}\cdot d\vec{u}$

$$0 = \frac{1}{2}\frac{d(u^2)}{dt}f(u) + f'(u)\,u\,\frac{1}{2}\frac{d(u^2)}{dt}$$

$$\Rightarrow 0 = \frac{1}{2}\frac{d(u^2)}{dt}\left[f(u) + u f'(u)\right]$$

but $f(u)+uf'(u) = \gamma + \gamma^2\frac{u^2}{c^2} \neq 0 \Rightarrow \frac{d(u^2)}{dt}=0$

Hence the particle moves with <u>constant speed</u> just as in the non-relativistic case. We may then write

$$\frac{d}{dt}\vec{P} = \frac{d}{dt}(m\gamma\vec{u}) = m\gamma\frac{d\vec{u}}{dt} = q\frac{\vec{u}}{c}\times\vec{B}$$

<u>or</u>

$$\frac{d\vec{u}}{dt} = \frac{q}{m^* c}\, \vec{u} \times \vec{B} \qquad \text{where } m^* = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}} = m\gamma$$

is sometimes referred to as the "relativistic mass".

For a uniform magnetic field, the particle moves on a circle (helix more generally) with angular velocity

$$\vec{\Omega} = -\frac{q\vec{B}}{m^* c} \qquad \text{(cyclotron frequency)}$$

with radius $R = \dfrac{u}{|\vec{\Omega}|}$ for velocity $\vec{u} \perp \vec{B}$ – i.e. no motion along $\vec{B}$.

Thus

$$R = \frac{m^* u c}{|q|\, B} = \frac{pc}{|q|\, B} = \frac{\sqrt{E^2 - m^2 c^4}}{|q|\, B}$$

where E is the energy and m is the rest mass.

<u>Homework:</u> Determine the motion of a charged particle in constant uniform electric and magnetic fields which are perpendicular to one another. Choose a coordinate system with $\vec{E} = E\hat{x}$ and $\vec{B} = B\hat{y}$. Take initial conditions

$x(0) = 0$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 0$, $\dot{z}(0) = v$.