Cherenkov Radiation (also Čerenkov)

is the electromagnetic analog of a sonic boom that occurs when a charged particle travels through a dielectric medium faster than the phase velocity of light in that medium.

To begin, we assume a frequency independent dielectric constant ϵ and take $\mu = 1$. In this section, the speed of light in vacuum is c_0 and the speed of light in the material is

$$c = \frac{c_0}{n} = \frac{c_0}{\sqrt{\epsilon}}$$ where n is the index of refraction.

Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi \rho}{\epsilon} \qquad \left(\vec{\nabla} \cdot \vec{D} = 4\pi\rho \text{ with } \vec{D} = \epsilon \vec{E} \right)$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c_0} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c_0} \vec{J} + \frac{\epsilon}{c_0} \frac{\partial \vec{E}}{\partial t} \qquad \left(\vec{\nabla} \times \vec{H} = \frac{4\pi}{c_0} \vec{J} + \frac{1}{c_0} \frac{\partial \vec{D}}{\partial t} \right)$$

$$\mu = 1 \Rightarrow \vec{B} = \vec{H}$$

where all fields are functions of space and time $(\vec{r}, t)$.

The $\vec{E}$ and $\vec{B}$ fields are obtained from potentials

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\frac{1}{\epsilon}\vec{\nabla}\Phi - \frac{1}{c_0}\frac{\partial \vec{A}}{\partial t}$$

The Ampere-Maxwell law becomes

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c_0}\vec{J} - \frac{\epsilon}{c_0}\left[\frac{1}{\epsilon}\frac{\partial \vec{\nabla}\Phi}{\partial t} + \frac{1}{c_0}\frac{\partial^2 \vec{A}}{\partial t^2}\right]$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c_0}\frac{\partial \Phi}{\partial t}\right) - \nabla^2\vec{A} = \frac{4\pi}{c_0}\vec{J} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$c = \frac{c_0}{n} = \frac{c_0}{\sqrt{\epsilon}}$$

We choose to work in Lorenz gauge:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c_0}\frac{\partial \Phi}{\partial t} = 0$$

Then

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = -\frac{4\pi}{c_0}\vec{J}(\vec{r},t)$$

Gauss' law gives

$$\vec{\nabla}\cdot\left(-\frac{1}{\epsilon}\vec{\nabla}\Phi - \frac{1}{c_0}\frac{\partial \vec{A}}{\partial t}\right) = \frac{4\pi\rho}{\epsilon}$$

$$\Rightarrow -\frac{1}{\epsilon}\nabla^2\Phi - \frac{1}{c_0}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \frac{4\pi\rho}{\epsilon}$$

or, using the Lorenz gauge condition

$$-\frac{1}{\epsilon}\nabla^2\Phi + \frac{1}{c_0^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{4\pi\rho}{\epsilon}$$

$$\Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\vec{r},t) = -4\pi\rho(\vec{r},t)$$

Thus we could simply take everything over from the vacuum case by carefully keeping track of where c and c_0 go (c_0 just goes with $\vec{J}$).

But we will derive the answer directly.

First we find

$$[\vec{\tilde{B}}_\omega]_{rad} = [\vec{\nabla}\times\vec{\tilde{A}}_\omega]_{rad}$$

where $\vec{\tilde{A}}_\omega(\vec{r})$ is the Fourier time transform of $\vec{A}(\vec{r},t)$.

For the current density, we take

$$\vec{J}(\vec{r},t) = q\,u\,\delta(x)\,\delta(y)\,\delta(z-ut)\,\hat{z}$$

$$\vec{\tilde{J}}_\omega(\vec{r}) = q\,u\,\delta(x)\,\delta(y)\int_{-\infty}^{\infty} dt\, e^{i\omega t}\,\delta(z-ut)\,\hat{z}$$

$$= q\,\delta(x)\,\delta(y)\,e^{i\omega\frac{z}{u}}\,\hat{z}$$

Then $$\vec{\tilde{A}}_\omega(\vec{r}) = \frac{1}{c_0}\iiint dV' \, \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, \vec{\tilde{J}}_\omega(\vec{r}')$$

which becomes in the far field

$$\vec{\tilde{A}}_\omega(\vec{r}) \longrightarrow \frac{1}{c_0}\,\frac{e^{i\frac{\omega}{c}r}}{r}\iiint dV' \, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}'}\,\vec{\tilde{J}}_\omega(\vec{r}')$$

So that

$$\left[\vec{\tilde{B}}_\omega\right]_{rad} = \frac{1}{c_0}\, i\frac{\omega}{c}\,\frac{e^{i\frac{\omega}{c}r}}{r}\,\hat{n}\times\hat{z}\iiint dV' \, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}'}\,\vec{\tilde{J}}_\omega(\vec{r}')$$

$$= \frac{q}{c_0}\, i\frac{\omega}{c}\,\frac{e^{i\frac{\omega}{c}r}}{r}\,\hat{n}\times\hat{z}\int_{z=-\infty}^{\infty} dz' \, e^{i\omega\left(\frac{1}{u}-\frac{\cos\theta}{c}\right)z'}$$

where $\hat{n}\cdot\hat{z} = \cos\theta$ and $|\hat{n}\times\hat{z}| = \sin\theta$.

Next we calculate the total energy radiated:

$$W = \frac{c_0}{4\pi}\int d\Omega \int_{-\infty}^{\infty} dt \, |\vec{B}_{rad}|^2 \, |\vec{r}-\vec{r}(t')|^2$$

which by Fourier transformation can be written as

$$W = \frac{q^2}{4\pi c_0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{c^2} \int d\Omega \sin^2\theta \left| \int_{-\infty}^{\infty} dz' e^{i\omega\left(\frac{1}{u} - \frac{\cos\theta}{c}\right)z'} \right|^2$$

Replace the $\pm\infty$ limits on the z' integral by $\pm\frac{L}{2}$:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dz' e^{i\omega\left(\frac{1}{u} - \frac{\cos\theta}{c}\right)z'} = \frac{2\sin\left[\omega\left(1 - \frac{u}{c}\cos\theta\right)\frac{L}{2u}\right]}{\omega\left(1 - \frac{u}{c}\cos\theta\right)\frac{1}{u}}$$

$$\Rightarrow W = 4\int_{-\infty}^{\infty} d\omega \frac{q^2}{4\pi c_0} \frac{\omega^2}{c^2} \int_{-1}^{+1} d\cos\theta \sin^2\theta \left[\frac{\sin^2\left[\left(1 - \frac{u}{c}\cos\theta\right)\frac{\omega L}{2u}\right]}{\frac{\omega^2}{u^2}\left(1 - \frac{u}{c}\cos\theta\right)^2}\right]$$

In the limit $L \to \infty$ the factor $[\dots]$ in the integrand is highly peaked around $1 - \frac{u}{c}\cos\theta = 0$ or around $\theta = \theta_c \equiv \arccos\left(\frac{c}{u}\right)$ where θ_c is the Cherenkov angle. Note that this angle is real, provided $c < u$ or $n > \frac{c_0}{u}$.

Because $[\cdots]$ is a rapidly varying function of θ while $\sin^2\theta$ is slowly varying, we replace $\sin^2\theta$ by $\left(1-\frac{c^2}{u^2}\right)$. Then defining $x \equiv \cos\theta$, the only important part of $[\cdots]$ comes from $x \approx \frac{c}{u}$. Thus we may extend the integration limits on x from $\int_{-1}^{+1}$ to $\int_{-\infty}^{+\infty}$ with negligible error as $L \to \infty$. Thus we write

$$W = \int_{\omega=0}^{\infty} W(\omega)\, d\omega \quad \text{or} \quad \int_{\omega=0}^{\infty} \frac{dW}{d\omega}\, d\omega$$

with
$$W(\omega) = \frac{8q^2}{4\pi\epsilon_0}\left(\frac{\omega}{c}\right)^2\left(1-\frac{c^2}{u^2}\right)\int_{x=-\infty}^{\infty} dx\, \frac{\sin^2\left[\left(1-\frac{u}{c}x\right)\frac{\omega L}{2u}\right]}{\frac{\omega^2}{u^2}\left(1-\frac{u}{c}x\right)^2}$$

Change the integration variable to

$$\eta \equiv \left(1-\frac{ux}{c}\right)\frac{\omega L}{2u}$$

So that

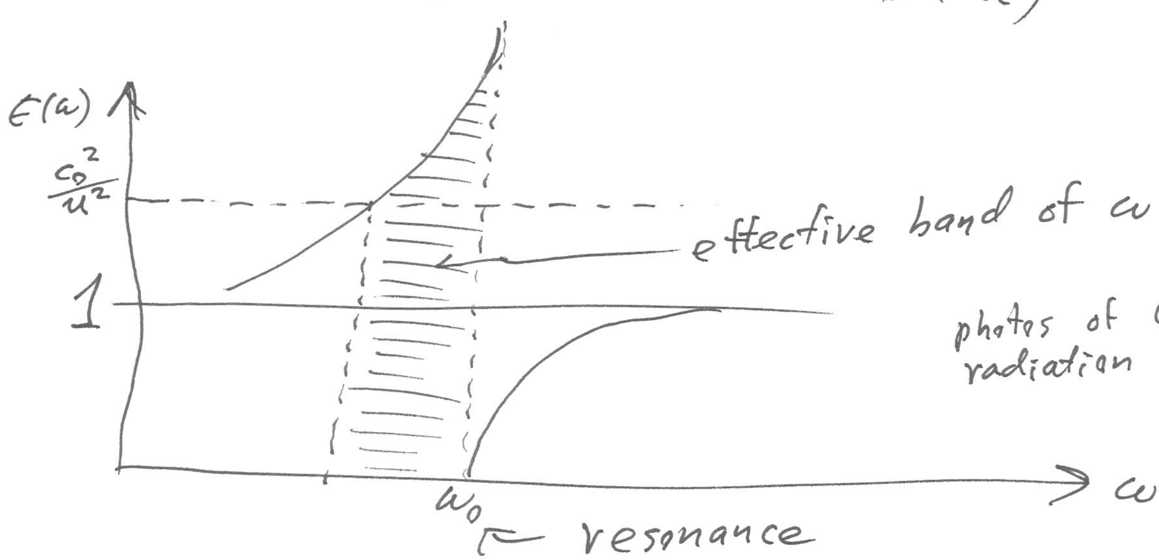
$$W(\omega) = \frac{2q^2}{\pi c_0}\left(\frac{\omega}{c}\right)^2\left(1-\frac{c^2}{u^2}\right)\frac{cL}{2\omega}\underbrace{\int_{\eta=-\infty}^{\infty} d\eta\,\frac{\sin^2\eta}{\eta^2}}_{\pi}$$

$$W(\omega) = \frac{q^2}{c_0}\left(\frac{\omega}{c}\right)\left(1-\frac{c^2}{u^2}\right)L$$

Note that $W(\omega) = 0$ for $u < c$.

While it appears that $W(\omega)$ is linear in ω, that is not the case because the index of refraction is really frequency dependent (a fact that we ignored until now).

$$\epsilon(\omega) = [n(\omega)]^2 \implies \epsilon(\omega) > \left(\frac{c_0}{u}\right)^2 \quad \text{for Cherenkov radiation}$$



photos of Cherenkov radiation are blue

The final form is then

$$\frac{W(\omega)}{L} = \frac{dW(\omega)}{dz} = \frac{q^2}{c_0}\frac{\omega}{c}\left(1 - \frac{c_0^2}{\epsilon(\omega)\,u^2}\right)$$

Thus $\frac{dW(\omega)}{dz}\,d\omega$ = energy loss per unit length of particle travel in the frequency range between ω and $\omega + d\omega$.

For a narrow range of ω, measurement of the Cherenkov angle gives a measure of particle speed.

$$\langle \theta_c \rangle = \left\langle \arccos\left(\frac{c}{u}\right) \right\rangle$$

and the angle brackets indicate average over a frequency band.