1. Show that the physical (real) electric field $\vec{\mathcal{E}}(\vec{r},t)$ is constrained by

$$\sum_{i=1}^{2}\sum_{j=1}^{2} A_{ij}\mathcal{E}_i\mathcal{E}_j = 1 \quad .$$

   This is a sum over polarizations. Find the 2×2 matrix $\mathbf{A}$ and show that the electric field traces out an ellipse as a function of time at each point in space $\vec{r}$.

2. (a) Show that the column vector $\vec{v}_{(1)}$ (defined in lecture) describes an ellipse with major axis along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and minor axis along $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eccentricity e. Show that the column vector $\vec{v}_{(2)}$ describes an ellipse with major axis along $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and minor axis along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eccentricity e.

   (b) Show that $\vec{v}^{\dagger}_{(i)} \cdot \vec{v}_{(j)} = \delta_{ij}$.

   (c) Show that $\mathbf{R}^{\dagger}\mathbf{R} = \mathbf{I}$ and hence that $\vec{u}^{\dagger}_{(i)} \cdot \vec{u}_{(j)} = \delta_{ij}$.

   (d) What geometric figure do the vectors $\vec{u}_{(i)}$ describe?

   (e) Write the density matrix ρ as a single matrix.

3. (a) Can the matrix $\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ be a density matrix?

   (b) Consider the following density matrices:

$$\rho_\alpha = \begin{pmatrix} +\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \end{pmatrix} \qquad\qquad \rho_\beta = \begin{pmatrix} +\frac{1}{4} & +\frac{i}{4} \\ -\frac{i}{4} & +\frac{3}{4} \end{pmatrix} \quad .$$

   Find the eigenvalues, eigenvectors, polarizations, and the nature of the two independent polarization states for each of the density matrices. (You may find the previous problem useful for this.)

BONUS (due when the homework is due):

A small loop of wire (radius a) lies a distance d above the center of a large loop (radius b). The planes of the two loops are parallel, and perpendicular to a common axis.

i. Suppose current I flows in the big loop. Find the magnetic flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant over the area of the little loop.)

ii. Suppose current I flows in the little loop. Find the magnetic flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)

iii. Find the mutual inductances and confirm that $L_{12} = L_{21}$.