30 November 95

B.) Energy and Momentum

Consider a system of charges characterized by a time-dependent charge density $\rho(\vec{r},t)$ and current density $\vec{J}(\vec{r},t)$. We assume that the only forces are electromagnetic (we can include other forces later). Then only the electric field does mechanical work — $\vec{v}\times\vec{B}$ is perpendicular to $\vec{v}$. The rate at which mechanical work is done is:

$$\frac{dW_{mech}}{dt} = \sum_{i=1}^{N} q_i\, \vec{v}_i \cdot \vec{E}(\vec{r},t)$$

$$\xrightarrow[\text{limit}]{\text{continuum}} \int_V dV\; \vec{J}(\vec{r},t)\cdot\vec{E}(\vec{r},t)$$

but a Maxwell relation can be inverted as

$$\vec{J} = \frac{c}{4\pi}\,\vec{\nabla}\times\vec{H} - \frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}$$

$$\therefore\ \frac{dW_{mech}}{dt} = \int_V dV\,\vec{J}\cdot\vec{E} = \frac{1}{4\pi}\int_V dV\left[c\,\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right]$$

Now write $\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{H})$

with the definition

$$\frac{dU_{field}}{dt} \equiv \frac{1}{4\pi} \int_V dV \left[\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right]$$

for the time rate of change of the field energy:

$$\frac{dW_{mech}}{dt} + \frac{dU_{field}}{dt} = -\frac{c}{4\pi} \int dV \, \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

Normally, we would use the divergence theorem to write the right-hand side as a surface integral, then we would let the surface go out to infinity and drop the term. We will see shortly that we cannot justify this procedure.

In a small time δt we have

$$\delta U_{field} = \frac{dU_{field}}{dt} \delta t = \frac{1}{4\pi} \int dV \left[\vec{E} \cdot \delta \vec{D} + \vec{H} \cdot \delta \vec{B} \right]$$

This form is valid for all types of media — para, dia, <u>and</u> ferro magnetic and electric materials.

If we specialize to <u>linear</u> media (no ferro materials) even if they are <u>anisotropic</u>

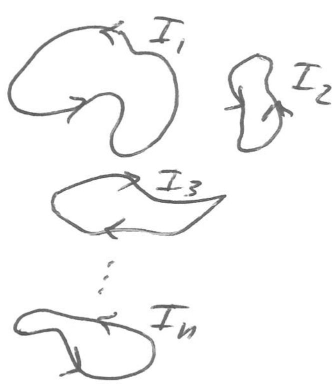
$$D_i = \sum_{j=1}^{3} \epsilon_{ij} E_j \qquad\qquad B_i = \sum_{j=1}^{3} \mu_{ij} H_j$$

then we can write the change symmetrically

$$\vec{H} \cdot \delta\vec{B} = \mu \vec{H} \cdot \delta\vec{H} = \frac{\mu}{2}\,\delta(H^2) = \delta\left(\frac{\vec{B}\cdot\vec{H}}{2}\right)$$

$$U_{electric} = \frac{1}{8\pi}\int dV\, \vec{E}\cdot\vec{D} \qquad\qquad U_{magnetic} = \frac{1}{8\pi}\int dV\, \vec{B}\cdot\vec{H}$$

For a while, we will concentrate on the magnetic energy. Suppose we have a system of N current carrying wires which are closed circuits (they will contain sources of EMF to maintain the currents).



We will see below that the energy of this system of currents can be written as

$$U_{mag} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij} I_i I_j$$

where L_{ij} is the inductance matrix.

The derivation is:

$$U_{mag} = \frac{1}{8\pi}\int dV\, \vec{H}\cdot\vec{B} = \frac{1}{8\pi}\int dV\, \vec{H}\cdot(\vec{\nabla}\times\vec{A})$$

$$= \frac{1}{8\pi}\int dV\, \vec{A}\cdot(\vec{\nabla}\times\vec{H}) + \underbrace{\frac{1}{8\pi}\int dV\, \vec{\nabla}\cdot(\vec{A}\times\vec{H})}_{0 \text{ by divergence theorem}}$$

the last term can be written as a surface integral. We can throw this term away as $S\to\infty$, unlike

$\int dV\, \vec{\nabla}\cdot(\vec{E}\times\vec{H})$ which we must keep.

We assume that there is no electric field due to the N currents, then $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}$ (no displacement current). $\Rightarrow \vec{\nabla}\times\vec{B} = \frac{4\pi\mu}{c}\vec{J}$

$$U_{mag} = \frac{1}{2c}\int dV\, \vec{J}_{(\vec{r})}\cdot\vec{A}_{(\vec{r})}$$

where $\vec{A}(\vec{r}) = \mu\int dV'\, \frac{\vec{J}(\vec{r}\,')}{c}\,\frac{1}{|\vec{r}-\vec{r}\,'|}$

Now at least we have an expression involving two currents

$$U_{mag} = \frac{\mu}{2c^2}\int dV\int dV'\, \frac{\vec{J}(\vec{r})\cdot\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Suppose the j^{th} wire has a uniform cross-sectional area S_j and uniform current density $J_j = \frac{I_j}{S_j}$.

$$\vec{J}(\vec{r}) = \sum_{i=1}^{N} \vec{J}_i(\vec{r})$$

$$\vec{A}(\vec{r}) = \mu \sum_{j=1}^{N} J_j \oint_{C_j} \frac{d\vec{r}_j'}{c} \int dS' \frac{1}{|\vec{r}-\vec{r}'|}$$

where C_j is the j^{th} closed circuit.

$$U_{mag} = \frac{\mu}{2c^2} \sum_{i=1}^{N}\sum_{j=1}^{N} \int \frac{dS}{S_i} \int \frac{dS'}{S_j} \oint_{C_i} \oint_{C_j} \frac{d\vec{r}_i \cdot d\vec{r}_j'}{|\vec{r}_i - \vec{r}_j'|} I_i I_j$$

So we can read the inductance matrix as

$$L_{ij} = \frac{\mu}{2c^2} \int \frac{dS}{S_i} \int \frac{dS'}{S_j} \oint_{C_i} \oint_{C_j} \frac{d\vec{r}_i \cdot d\vec{r}_j'}{|\vec{r}_i - \vec{r}_j'|}$$

L_{ii} are called <u>self-inductances</u> and L_{ij} for $i \neq j$ are <u>mutual inductances</u>, sometimes given the symbol M_{ij}.

[Remember: $C_{ij} = -\oint_{S_i} \oint_{S_j} \frac{dS_i}{4\pi} \frac{dS_j'}{4\pi} \hat{n}_i \cdot \vec{\nabla}\, \hat{n}_j \cdot \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|}$]
for capacitance

There is an alternative way to calculate the inductance coefficients which is useful in cases with a lot of symmetry. Return to the expression for the magnetic energy of a system of currents embedded in a material of uniform permeability μ.

$$U_{mag} = \frac{1}{8\pi}\int dV\, \vec{B}\cdot\vec{H} = \frac{1}{8\pi\mu}\int dV\, B^2$$

The magnetic field at point $\vec{r}$ is

$$\vec{B}(\vec{r}) = \sum_{i=1}^{N} \vec{B}_i(\vec{r})$$ where $\vec{B}_i(\vec{r})$ is the magnetic induction at $\vec{r}$ due to the i^{th} current.

$$U_{mag} = \frac{1}{8\pi\mu}\sum_{i=1}^{N}\sum_{j=1}^{N}\int dV\, \vec{B}_i\cdot\vec{B}_j$$

multiply and divide by the currents $I_i I_j$

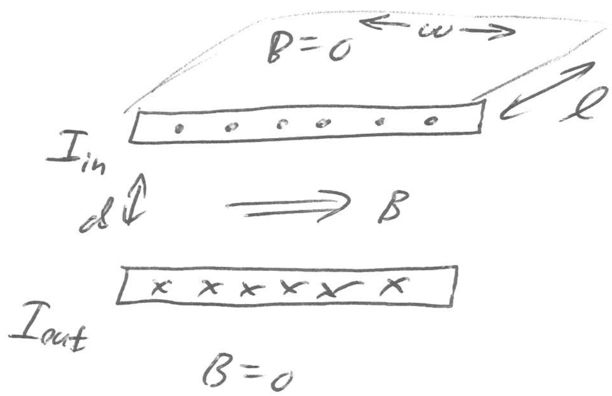
$$= \frac{1}{2}\sum_i\sum_j I_i I_j \left[\frac{1}{4\pi\mu I_i I_j}\int dV\, \vec{B}_i\cdot\vec{B}_j\right]$$

and we read off

$$L_{ij} = \frac{\int dV\, \vec{B}_i\cdot\vec{B}_j}{4\pi\mu\, I_i I_j}$$ (try Jackson 6.3)

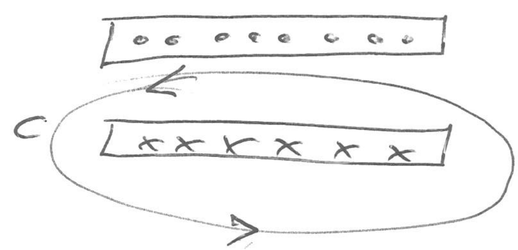
A simple example:

2 current-carrying plates with length ℓ, width w, separated by distance d.



Use Ampere's law. Surround one plate by a mathematical curve C.

B is non-zero only between the plates.



$$\oint_C d\vec{\ell}\cdot\vec{B} = \frac{4\pi}{c} I_{enc}$$

$$wB = \frac{4\pi}{c} I \quad \Rightarrow \quad B = \frac{4\pi}{cw} I$$

energy density $\dfrac{B^2}{8\pi} = \dfrac{1}{2}\dfrac{4\pi}{c^2}\dfrac{I^2}{w^2}$

$$U_{mag} = \int dV \frac{B}{8\pi} = V\frac{B^2}{8\pi} = (w\ell d)\frac{1}{2}\frac{4\pi}{c^2}\frac{I^2}{w^2} = \frac{1}{2}LI^2$$

where the inductance is

$$L = \frac{4\pi}{c}\frac{\ell d}{w} \quad (\ell \gg w \gg d) = L_{11}$$

self-inductance of the one and only conductor

Returning now to the energy balance equation

$$\left(\frac{dW_{mech}}{dt}\right) + \left(\frac{dU_{field}}{dt}\right) = -\frac{c}{4\pi}\int_V dV\, \vec{\nabla}\cdot(\vec{E}\times\vec{H})$$

we interpret the terms as follows:

$\frac{dW_{mech}}{dt}$ is the rate at which work is done on the <u>free</u> charges in volume V. This is equal to the rate of change of mechanical energy of the free charges.

$\frac{dU_{field}}{dt}$ is the rate at which the energy stored in the electromagnet field is changing.

The left-hand side of the energy balance equation is the rate at which the total energy of the system (free charges + field) is changing.

Note that the mechanical energy of the <u>bound</u> charges and currents is included in the <u>field</u> energy through the polarization and magnetization relations:

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$
$$\vec{B} = \vec{H} + 4\pi\vec{M}$$

Using the divergence theorem, we can write the right-hand side as

$$-\oint_S dS\ \hat{n}\cdot\vec{S} \qquad \text{where } \vec{S} \equiv \frac{c}{4\pi}\vec{E}\times\vec{H}$$

is called the Poynting vector. Usually we would allow the surface to grow to infinite radius and argue that the fields in the integrand vanish fast enough as $r\to\infty$, and for static fields this is true:

$dS = d\Omega\ R^2$

$E_{static} \sim \frac{1}{R^2}$ or faster

$H_{static} \sim \frac{1}{R^3}$ or faster (no monopole $\frac{1}{R^2}$ contribution)

$\Rightarrow \oint dS\ \hat{n}\cdot\vec{S} \sim \frac{1}{R^3}$ or faster

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But there are radiating solutions to Maxwell's equations (the entire second part of the course) which only fall off as $\frac{1}{R}$ as $R\to\infty$

$$\left.\begin{array}{l} dS \sim R^2 \\ E_{rad} \sim 1/R \\ H_{rad} \sim 1/R \end{array}\right\} \Rightarrow \oint dS\ \hat{n}\cdot\vec{S} \text{ is non-zero!}$$

We obtain a finite result for the right-hand side even for an infinite surface.

In the case of radiation fields, $\vec{S}$ represents an <u>energy flux</u> (energy per unit area per unit time).

For finite regions, we may interpret the left-hand side: $\left[\frac{dW_{mech}}{dt} + \frac{dU_{field}}{dt}\right]_V$ as the rate of change of total energy within V, and the right-hand side $-\oint_S dS\, \hat{n}\cdot\vec{S}$ as the rate at which energy flows <u>into</u> (because of the minus sign) the region bounded by the closed surface.

The interpretation of $\vec{S}$ as an energy flux can be misleading and appears absurd at times. For example, a charged particle sitting <u>at rest</u> in a <u>static</u> magnetic field will have a non-zero Poynting vector. Thus according to our interpretation, there would seem to be a flow of electromagnetic energy even though all fields are <u>static</u>! But the only quantity of physical significance is the surface integral $\oint_S dS\, \hat{n}\cdot\vec{S}$. This surface integral will vanish for the example above.

For static fields <u>in the absence of currents</u>

$$\vec\nabla \cdot (\vec E \times \vec H) = \underbrace{(\vec\nabla \times \vec E)}_{0} \cdot \vec H - \underbrace{(\nabla \times \vec H)}_{0} \cdot \vec E = 0$$

For static fields in the presence of currents, the Poynting vector provides an influx of energy sufficient to replace the energy lost through Joule heating. This energy must flow <u>in</u> since the electric and magnetic fields are maintained in the region of the currents and hence the increase in mechanical energy (Joule heating of the atoms) cannot come at the expense of the fields. To see how this works in a special case, consider a long straight wire carrying current I. Then outside the wire at the surface:

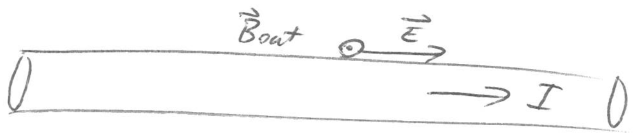
$$\vec B = \frac{2I}{ca}\,\hat e_\varphi = \vec H$$

$$\vec E = J/\sigma \quad \hat e_z$$

a = radius of wire

σ = conductivity

$J = I/\pi a^2$ = uniform current density



Remember the tangential component of $\vec E$ is continuous across a boundary.

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H} = \frac{1}{2\pi}\frac{IJ}{\sigma a}\,\hat{e}_z\times\hat{e}_\varphi = -\frac{1}{2\pi}\frac{I^2}{\pi a^2\sigma}\frac{1}{a}\hat{e}_\rho$$

$$\Rightarrow -\oint_S ds\,\hat{n}\cdot\vec{S} = \frac{2\pi a\ell}{2\pi a}\frac{I^2}{\pi a^2\sigma} = +\frac{\ell}{\pi a^2\sigma}I^2 = +I^2R$$

includes end pieces where $\hat{n}\cdot\vec{S} = 0$

outward normal

power <u>into</u> wire

R = resistance of segment of wire of length ℓ.

This is the expression for power loss in a resistive circuit.

The flux of electromagnetic energy is <u>into</u> the wire. This energy maintains a steady current and is converted into heat (mechanical energy).

— End lecture # 25 —