Poynting Vector:

$$\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

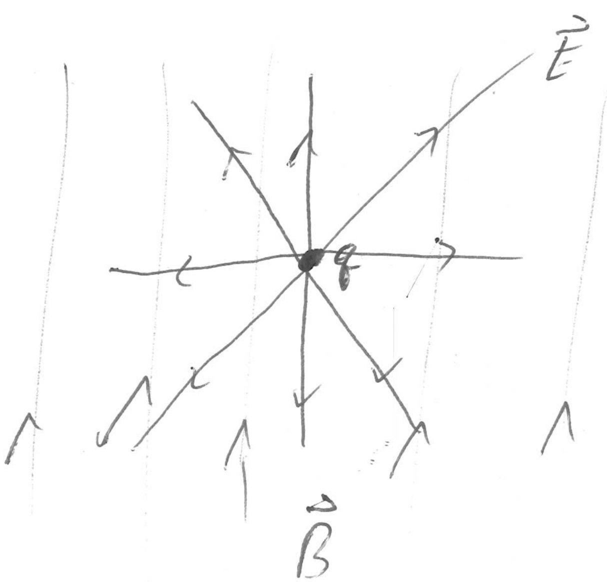
energy per unit time
per unit area
transported by the
fields.

$$\frac{dW_{mech}}{dt} + \frac{d}{dt} \iiint_V \underbrace{\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)}_{u_{em} \left\{ \substack{\text{energy density of} \\ \text{electromagnetic field}} \right.} dV = -\oint\!\!\!\oint_S \vec{S} \cdot d\vec{a}$$

This is the integral form of Poynting's Theorem.

The differential form is obtained using the divergence theorem (Green's thm, Gauss thm).

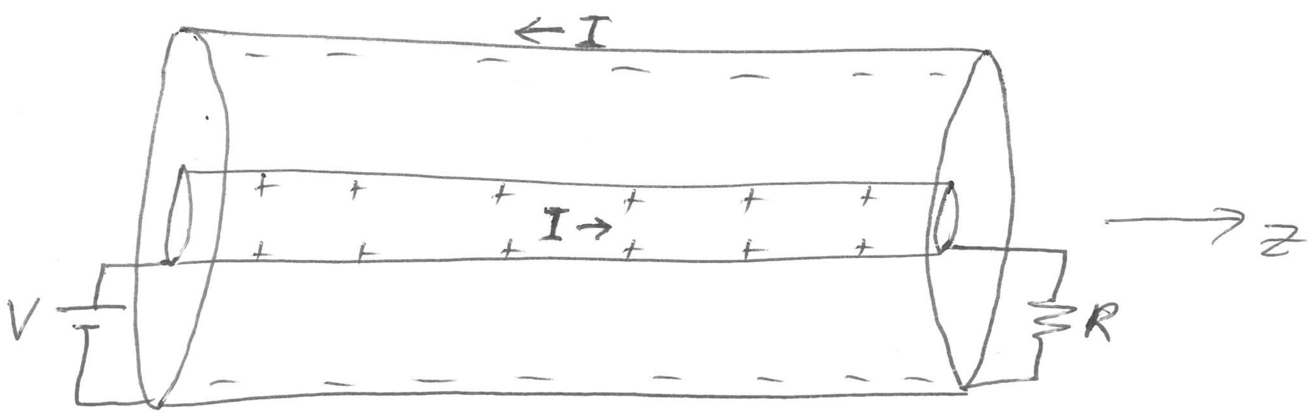
$$\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) = - \vec{\nabla} \cdot \vec{S}$$



$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \neq 0$$

Another example

A long coaxial cable of length l consists of an inner conductor of radius a and an outer conductor of radius b. It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length $+\lambda$ and a steady current I to the right; the outer conductor carries opposite charge and current. Find the electromagnetic energy flow.



For $\rho < a$ and $\rho > b$, both $\vec{E}$ and $\vec{B}$ are zero.

For $a < \rho < b$, Gauss' Law give the $\vec{E}$ field:

$$\oiint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E\, 2\pi\rho l = \frac{\lambda l}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 \rho}\,\hat{\rho}$$

Ampere's Law gives the $\vec{B}$ field:

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \implies B\, 2\pi s = \mu_0 I$$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi} \quad , \; a < s < b$$

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{z}$$

energy is flowing to the right. The power (energy/time) transported is

$$P = \iint \vec{S}\cdot d\vec{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_{s=a}^{b}\frac{1}{s^2}\, s\,ds\int_{\varphi=0}^{2\pi} d\varphi = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) = IV$$

The electromagnetic field also carries linear momentum and angular momentum in addition to energy.

In Quantum Electrodynamics, the photon carries energy $\hbar\omega$, linear momentum $\frac{\hbar\omega}{c}$, and angular momentum $\hbar$ (spin-1).

Field Momentum

Force on charge q: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Force on charge distribution in volume V:

$$\vec{F} = \iiint_V \rho(\vec{r}) \left[\vec{E}(\vec{r}) + \vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] dV$$

$$= \iiint_V \left[\rho(\vec{r}) \vec{E}(\vec{r}) + \vec{J}(\vec{r}) \times \vec{B}(\vec{r}) \right] dV$$

The "force density" or force per unit volume is

$$\vec{f}(\vec{r}) = \rho(\vec{r}) \vec{E}(\vec{r}) + \vec{J}(\vec{r}) \times \vec{B}(\vec{r})$$

Use the Maxwell Equations to eliminate the sources in terms of the fields:

$$\vec{f} = \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

simplify using

$$\frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t}$$

and Faraday's Law $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$

so $$\frac{\partial \vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

and now the force density is

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] - \frac{1}{\mu_0}\left[\vec{B}\times(\vec{\nabla}\times\vec{B})\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

We can make these terms look symmetric by adding $(\vec{\nabla}\cdot\vec{B})\vec{B}$ since $\vec{\nabla}\cdot\vec{B} = 0$.

Also notice

$$\vec{\nabla}(E^2) = \vec{\nabla}(\vec{E}\cdot\vec{E}) = 2(\vec{E}\cdot\vec{\nabla})\vec{E} + 2\vec{E}\times(\vec{\nabla}\times\vec{E})$$
$$\vec{\nabla}(B^2) = \vec{\nabla}(\vec{B}\cdot\vec{B}) = 2(\vec{B}\cdot\vec{\nabla})\vec{B} + 2\vec{B}\times(\vec{\nabla}\times\vec{B})$$

So

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$
$$-\frac{1}{2}\vec{\nabla}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\mu_0\frac{\partial}{\partial t}\frac{\vec{E}\times\vec{B}}{\mu_0}$$

this is $\vec{S}$

Maxwell stress tensor – rank 2

$$T_{ij} \equiv \epsilon_0\left(E_i E_j - \frac{\delta_{ij}}{2}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{\delta_{ij}}{2}B^2\right)$$

this is the force per unit area (stress) acting on a surface. Diagonal elements are "pressures"; off-diagonal elements are "shears".

$$f_j = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} T_{ij} - \epsilon_0 \mu_0 \frac{\partial}{\partial t}\vec{S}$$

e.g. $T_{11} = T_{xx} = \frac{\epsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right)$

$T_{12} = T_{xy} = \epsilon_0 E_x E_y + \frac{B_x B_y}{\mu_0}$

Newton's 2^{nd} Law : $\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \iiint_V \vec{f}(\vec{r})\, dV$

$$\left(\frac{d\vec{P}_{mech}}{dt}\right)_i + \left(\epsilon_0\mu_0 \frac{d}{dt}\iiint_V \vec{S}(\vec{r})\, dV\right)_i = \oiint_S \sum_{j=1}^{3} T_{ij}\, da_j$$

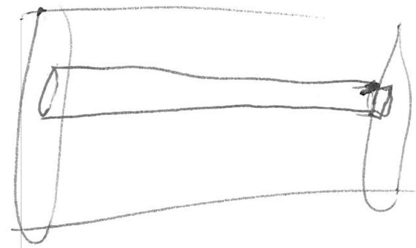
$\frac{d}{dt}\vec{P}_{em} = \frac{d}{dt}$(electromagnetic momentum)

Field Momentum

$$\vec{P}_{em} = \epsilon_0 \mu_0 \iiint_V \vec{S}(\vec{r})\, dV$$

↑
not a function
of position.

For our coax cable example:



$$\vec{P}_{em} = \frac{\mu_0 \lambda I \hat{z}}{4\pi^2}(2\pi)\ell \int_{s=a}^{b} \frac{1}{s^2}\, s\, ds = \frac{\mu_0 \lambda I \ell}{2\pi} \ln\left(\frac{b}{a}\right) \hat{z}$$

The coax cable is not moving and the $\vec{E}$ and $\vec{B}$ fields are static. Why is momentum flowing to the right? The center of mass of the system is at rest, so the total momentum must be zero. Where is the "hidden momentum"?