It is instructive to look at the dielectric function $\epsilon(\omega)$ as a mathematical object:

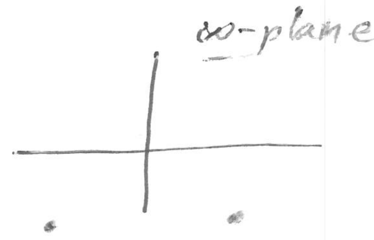
$$\epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$

by considering ω as a complex variable. Then $\epsilon(\omega)$ has a very simple analytic structure — it has poles at

$$\omega^2 - \omega_n^2 + i\gamma_n \omega = 0$$

or

$$\omega = \frac{-i\gamma_n}{2} \pm \sqrt{\omega_n^2 - \frac{\gamma_n^2}{4}}$$



The poles are in the lower half of the complex ω-plane. This behavior can be shown to follow from the principle of causality which, loosely speaking, means that a localized disturbance in space and time has effects in remote regions which are experienced with a time delay because of the finite velocity of light — that is, "cause" precedes "effect".

A consequence is that we have the famous Kramers-Kronig relation

$$\epsilon_R(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \epsilon_I(\omega')\, d\omega'}{\omega'^2 - \omega^2}$$

which connects the real and imaginary parts of $\epsilon(\omega)$ for real ω.

Thus dispersive aspects $\epsilon_R(\omega)$ and absorptive aspects $\epsilon_I(\omega)$ are linked together. There is an analogous connection between the real and imaginary parts of the forward scattering amplitudes in particle physics that was a popular subject in the late 1950's and early 1960's. (cf. Optical theorem.)

<u>Connection between analyticity and causality</u>

The dielectric function $\epsilon(\omega)$ has poles in the lower half ω-plane, so $\epsilon(\omega)$ is an analytic function of ω in the upper half ω-plane. We now demonstrate the connection between analyticity and causality.

Suppose we have a "cause" described by some function of time $c(t)$ and its "effect" described by $E(t)$. These are related by a response function $R(t)$ through

$$E(t) = \int_{-\infty}^{+\infty} R(t-t')\, c(t')\, dt'$$

[Remember $\underbrace{x(t)}_{\text{response}} = \int_{-\infty}^{+\infty} \underbrace{G(t,t')}_{\text{Green function}} \underbrace{F(t')}_{\text{forcing}} dt'$ from mechanics]

Introduce Fourier Transforms $X = E, R, C$.

$$X(t) = \int_{\omega=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} \tilde{X}(\omega)\, d\omega$$

$$\tilde{X}(\omega) = \int_{t=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{+i\omega t} X(t)\, dt$$

Then we find $\tilde{E}(\omega) = \tilde{R}(\omega)\, \tilde{C}(\omega)$

where we made use of the Dirac delta function

$$\delta(\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{+\infty} e^{i\omega t}\, dt$$

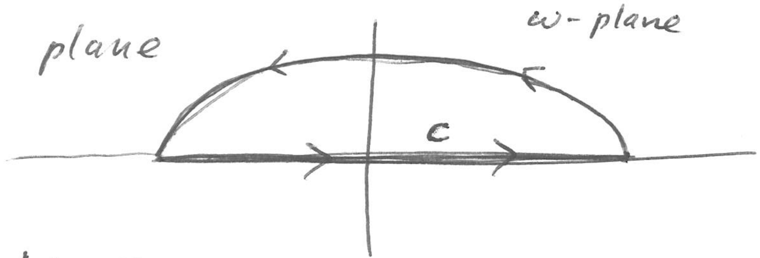
The statement of causality is that $R(t-t') = 0$ for $t < t'$. How can this property be ensured?

Suppose $\hat{R}(\omega)$ is analytic in the upper half plane and that $\tilde{R}(\omega) \to 0$ sufficiently fast as $|\omega| \to \infty$.

then for $t-t' < 0$ we may evaluate the integral

$$R(t-t') = \int_{\omega=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega(t-t')} \hat{R}(\omega)\, d\omega = \int_{\omega=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{+i\omega|t-t'|} \hat{R}(\omega)\, d\omega$$

by "closing the contour" in the upper half plane because on an infinite semicircle in the upper half plane



$\omega = \omega_R + i\,\omega_I$

$$e^{+i\omega(t-t')} = e^{+i\omega_R(t-t')} \cdot e^{-\omega_I(t-t')} \to 0 \quad \text{as} \quad \omega_I \to \infty.$$

Thus for $t < t'$

$$R(t-t') = \int_{\omega=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{+i\omega(t-t')} \tilde{R}(\omega)\, d\omega = \oint_C \frac{1}{\sqrt{2\pi}} e^{+i\omega(t-t')} \tilde{R}(\omega)\, d\omega = 0$$

because of <u>Cauchy's Theorem</u>: If $f(z)$ is analytic in region Ω, then $\oint_c f(z)\, dz = 0$ for any curve c in Ω. Thus analyticity in the upper half complex ω-plane implies causality.

Wave packets in a dispersive medium

Consider a medium in which absorption is negligible. Suppose the electric field is linearly polarized along $\vec{\epsilon}_1$ and the wave propagates along the z-axis.

Then $\vec{\mathcal{E}}(z,t) = \mathrm{Re}\, \vec{E}(z,t)$

where $\vec{E}(z,t)$ is now taken not as a plane wave but rather as a superposition of plane waves — that is a "wave packet"

$$\vec{E}(z,t) = \vec{\epsilon}_1\, E(z,t) = \vec{\epsilon}_1 \int_{k_z=-\infty}^{+\infty} \frac{dk_z}{\sqrt{2\pi}}\, \tilde{E}(k_z)\, e^{i[k_z z - \omega(k_z) t]}$$

the magnetic field will then be given by

$$\vec{B}(z,t) = \vec{\epsilon}_2\, B(z,t) = \vec{\epsilon}_2 \int_{k_z=-\infty}^{+\infty} \frac{dk_z}{\sqrt{2\pi}}\, \tilde{B}(k_z)\, e^{i[k_z z - \omega(k_z) t]}$$

where $\vec{\epsilon}_1 \times \vec{\epsilon}_2 = \hat{z}$ and $\tilde{B}(k_z) = \dfrac{k_z c}{\omega(k_z)}\, \tilde{E}(k_z)$

The frequency of the wave must be an even function of k_z: $\omega(-k_z) = \omega(k_z)$ for an isotropic medium.

We next assume that $\tilde{E}(k_z)$ is sharply peaked about some value k_0 which we take to be positive. We then make a Taylor expansion of $\omega(k_z)$ about k_0:

$$\omega(k_z) = \omega_0 + \omega_0'(k_z - k_0) + \frac{1}{2}\omega_0''(k_z-k_0)^2 + \ldots$$

where $\omega_0 \equiv \omega(k_0)$, $\omega_0' = \left[\frac{d\omega(k_z)}{dk_z}\right]_{k_z=k_0}$, $\omega_0'' = \left[\frac{d^2\omega(k_z)}{dk_z^2}\right]_{k_z=k_0}$

The quantity ω_0' is called the <u>group velocity</u> of the wave packet at wave number k_0.

For a dispersion<u>less</u> medium then

$$\omega = \frac{|k_z| c}{n}$$ where n = <u>constant</u> index of refraction

and $\omega' = \frac{c}{n} = v$ = <u>phase</u> <u>velocity</u> of the wave.

In such a dispersionless medium we have

$$E(z,t) = \int_{-\infty}^{+\infty} \frac{dk_z}{\sqrt{2\pi}} \tilde{E}(k_z)\, e^{i(k_z z - \omega t)}$$

$$= \int_0^{\infty} \frac{dk_z}{\sqrt{2\pi}} \tilde{E}(k_z)\, e^{ik_z(z-vt)} + \int_{-\infty}^{0} \frac{dk_z}{\sqrt{2\pi}} \tilde{E}(k_z)\, e^{ik_z(z+vt)}$$

$$\equiv E_+(z-vt) + E_-(z+vt)$$

right-moving $\uparrow k_z > 0 \longrightarrow$ $\qquad \uparrow k_z < 0$ left-moving $\longleftarrow$