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On the Galilean non-invariance of classical electromagnetism

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Abstract

When asked to explain the Galilean non-invariance of classical electromagnetism on the basis of pre-relativistic considerations alone, students—and sometimes their teachers too—may face an impasse. Indeed, they often argue that a pre-relativistic physicist could most obviously have provided the explanation 'at a glance', on the basis of the presence of a parameter c with the dimensions of a velocity in Maxwell's equations, being well aware of the fact that *any* velocity is non-invariant in Galilean relativity. This 'obvious' answer, however popular, is not correct due to the actual observer-invariance of the Maxwell parameter c in pre-relativistic physics too. A pre-relativistic physicist would therefore have needed a different explanation. Playing the role of this physicist, we pedagogically show how a proof of the Galilean non-invariance of classical electromagnetism can be obtained, resting on simple pre-relativistic considerations alone.

1. Introduction

In their introductory courses on special relativity, one of the very first things students are told about is that classical electromagnetism [1, 2] is not Galilean invariant—i.e., that the equational relationships between fields etc are not preserved under Galilean transformations. Maxwell's theory predicts the existence of electromagnetic waves travelling in vacuo at speed c,[3] and the Galilean non-invariance of velocities—*any* velocity—implies that Maxwell's equations in their simple form are valid only in a well-defined reference frame, identified as 'the luminiferous ether', a space-filling imponderable medium supporting the wave propagation itself. Hence,

[3] The symbol 'c' did not always have the current conventional meaning of 'speed of light in vacuo'. Originally, it was employed to indicate a ratio of electric units, and as such it was measured by Weber and Kohlrausch, who reported their result in [3] to which Maxwell made reference in his Treatise [1]. For a historical account of these issues see, e.g., the second and third parts of [3].

Galilean relativity predicts that an observer not at rest with respect to the ether measures a different speed of propagation for the electromagnetic waves. Put to the experimental test, this prediction was dramatically contradicted by the groundbreaking Michelson–Morley result [4], for the explanation of which a new ('Lorentz') kind of invariance [5–11] had to be introduced and was then interpreted by Einstein in terms of new physics [12].

This introductory approach may leave the students with the wrong impression that the reason why classical electromagnetism is not Galilean invariant rests ultimately in the presence of the parameter c in Maxwell's equations. In order to avoid this misconception, students may be pedagogically invited to take a different perspective. We can make them note that the parameter c appearing in Maxwell's equations can actually be regarded as a property of Newtonian free space itself[4], due to its very definition:

$$c \equiv \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{1.1}$$

(SI units), in terms of two free space quantities, namely the vacuum permittivity ε_0 and the vacuum permeability μ_0, which can be separately determined[5]. Since the vacuum in Newtonian physics is observer-invariant, both ε_0 and μ_0 can be regarded as observer-independent scalars, i.e., universal constants characterizing the vacuum in any reference frame. Hence, resting on definition (1.1), it follows that this same observer independence characterizes Maxwell's parameter c as well, which therefore behaves as a scalar invariant under frame transformations.

Although the above conclusion might seem contrived, relying as it apparently does on the specific system of physical units employed, the final result actually holds independently of it, as we shall see: the scalar invariance of Maxwell's parameter c does characterize *pre*-relativistic physics as well; the choice of SI units is simply instrumental in letting this fact emerge more transparently through the above argument. As a matter of fact, in order to let the following exposition be more 'student friendly', we shall employ these same pedagogically convenient SI units throughout the body of this paper; the due generalization to arbitrary units will be postponed to an apposite appendix, set at the end of this paper.

Once the observer-independent character of Maxwell's parameter c in pre-relativistic physics is recognized, students will realize that—contrary to the widespread opinion—it is *not* the presence of this parameter in Maxwell's equations which provides a patent clue to the Galilean non-invariance of classical electromagnetism. Of course, we know that Maxwell's theory is *Lorentz* invariant instead; but how could a 19th century physicist, unaware of Lorentz transformations, have proved the Galilean non-invariance of classical electromagnetism in a consistently pre-relativistic way?

This is what we are going to see.

2. The ingredients

Let us first review the basic ingredients our pre-relativistic physicist has at their disposal: the Galilean transformation laws for Newtonian mechanics; electric charge invariance; Maxwell's equations of classical electromagnetism.

[4] The role of c (and of the vacuum resistance Ω_0) as a fundamental property of *spacetime* (relativistic theory, of course) is discussed, e.g., in [13].

[5] Actually, the latter is by definition assigned the exact numerical value $\mu_0 = 4\pi \times 10^7$ N A^{-2} in SI units, while the numerical value of the former can be experimentally measured, e.g., by recurring to the Maxwell commutator bridge [1, 15–18]; thus, we find [18] that $\varepsilon_0 = (8.81 \times 10^{-12} \pm 1\%)$ F m^{-1}. The fact that the numerical value of c obtained from equation (1.1) using these data actually coincides, within measurement precision, with the experimentally determined speed of propagation of light in vacuo, lets us usually speak of Maxwell's constant c as 'the speed of light in vacuo' tout court. A broader meaning of c in Einsteinian relativity is discussed in [14].

2.1. Galilean relativity

Newtonian mechanics is invariant under the set of transformations

$$t' = t + a, \qquad \mathbf{x}' = R\mathbf{x} - \mathbf{v}_0 t + \mathbf{b}, \tag{2.1}$$

where $a \in \mathbb{R}$, $R \in SO(3)$ and $\mathbf{v}_0, \mathbf{b} \in \mathbb{R}^3$. The above set constitutes the general Galilean invariance group of Newtonian mechanics; in the following, we shall focus our attention—as commonly done—on the 'usual' Galilean subgroup given by

$$t' = t, \qquad \mathbf{x}' = \mathbf{x} - \mathbf{v}_0 t. \tag{2.2}$$

From these transformations, recalling the definitions

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}, \qquad \mathbf{v}' = \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'}, \tag{2.3}$$

we also have

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \tag{2.4}$$

as the Galilean transformation for the velocities. Since

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\left(\frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial t} + v_0^i \frac{\partial}{\partial x^i}, \tag{2.5}$$

$$\frac{\partial}{\partial x'^i} = \frac{\partial t}{\partial x'^i}\frac{\partial}{\partial t} + \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^i}, \tag{2.6}$$

we have

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla, \qquad \nabla' = \nabla, \tag{2.7}$$

as the Galilean transformation laws for the partial derivative operators. Transformations (2.2) do not allow for spatial contraction, nor for time dilation; hence, the spatial volume V is a Galilean invariant:

$$V' = V. \tag{2.8}$$

2.2. Electric charge invariance

A tenet of classical electromagnetism is electric-charge conservation under frame transformations. Hence, if $V = V(t)$ is the spatial volume where the Galilean invariant charge q is contained with the charge density $\varrho = \varrho(t, x(t))$, (2.8) assures that ϱ is Galilean invariant itself, i.e.:

$$\varrho' = \varrho. \tag{2.9}$$

Defining now the charge current density as

$$\mathbf{j} \equiv \varrho \mathbf{v} \tag{2.10}$$

and requiring relation (2.10) to be preserved under Galilean transformations, from (2.2) and (2.9) the Galilean transformation law for $\mathbf{j}$ follows as

$$\mathbf{j}' = \mathbf{j} - \varrho \mathbf{v}_0. \tag{2.11}$$

2.3. Maxwell's equations

In SI units, the four differential Maxwell's equations of classical electromagnetism in vacuo read

$$\nabla \cdot \mathbf{B} = 0, \tag{2.12}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.13}$$

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0}, \tag{2.14}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{2.15}$$

where **E** and **B** are the electric and magnetic fields and Maxwell's constant c is defined by equation (1.1).

3. Galilean transformations for the fields and invariance of Maxwell's equations

As explained in section 1, taking the point of view of a pre-relativistic observer we want to see from where the incompatibility between the classical electromagnetism and the Galilean relativity arises. To this end, our pre-relativistic physicist has first to obtain the Galilean transformations for the electric and magnetic fields. This is actually not a conceptually trivial task, since these fields are not emcompassed by Newtonian *mechanics*—which implies that these transformation laws have to be worked out from scratch in a fully pre-relativistic consistent way.

We are obviously not allowed to follow the usual procedure presented in many textbooks, namely to start from the *Lorentz* transformations for **E** and **B** and then take their nonrelativistic limit: indeed, our pre-relativistic physicist does not even know what the expression 'Lorentz transformations' means! Hence, the Galilean transformations for the fields must be derived from some other request, consistent with the Newtonian point of view; once this essential step is taken, the invariance properties of Maxwell's equations can subsequently be put to the test with Galilean relativity.

3.1. Pre-relativistic requirement of Lorentz force invariance

In order to obtain the Galilean transformations for the fields, a pre-relativistic physicist might have quite sensibly made reference to the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{3.1}$$

and required its Galilean invariance. Such a requirement would appear most natural, since the Lorentz force—the quantity actually measured by the observers in their reference frames—determines the acceleration, according to Newton's second law, and the acceleration is obviously Galilean invariant itself. From charge invariance, we have

$$\mathbf{F}' = \mathbf{F} \quad \Longrightarrow \quad \mathbf{E}' + \mathbf{v}' \times \mathbf{B}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{3.2}$$

from which, recalling (2.4), it follows that

$$\mathbf{E}' + \mathbf{v} \times (\mathbf{B}' - \mathbf{B}) = \mathbf{E} + \mathbf{v}_0 \times \mathbf{B}'. \tag{3.3}$$

The only possible solution of (3.3) not involving velocities in a specific reference frame, and without restrictions on the choice of $\mathbf{v}'$, is

$$\mathbf{B}'(\mathbf{x}', t') = \mathbf{B}(\mathbf{x}, t), \tag{3.4}$$

$$\mathbf{E}'(\mathbf{x}', t') = \mathbf{E}(\mathbf{x}, t) + \mathbf{v}_0 \times \mathbf{B}(\mathbf{x}, t), \tag{3.5}$$

where, for clarity, the dependence of the fields on the primed and unprimed coordinates has been indicated explicitly. Equations (3.4) and (3.5) provide the consistently pre-relativistic transformation rules for the fields guaranteeing the Galilean invariance of the Lorentz force (3.1). It is worth noting that these equations actually coincide with the nonrelativistic 'magnetic' limit of the Lorentz transformations for the fields, as defined in [19].

Armed with equations (2.7), (2.9), (2.11), (2.12)–(2.15), (3.4) and (3.5), our pre-relativistic physicist is now ready to put to the test the Galilean invariance of classical electromagnetism.

3.2. Maxwell's equations and Galilean invariance

3.2.1. Magnetic Gauss' law. From equations (2.7), (2.12) and (3.4) we immediately find

$$\nabla' \cdot \mathbf{B}' = \nabla \cdot \mathbf{B} = 0, \tag{3.6}$$

i.e., the solenoidal character of the magnetic field is Galilean invariant.

3.2.2. Faraday's law. From equations (2.7), (2.13), (3.4), (3.5) and the formulae in appendix A, recalling $\mathbf{v}_0 = \text{const.}$, we have

$$\begin{aligned}\nabla' \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'} &= \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) + \nabla \times (\mathbf{v}_0 \times \mathbf{B}) + (\mathbf{v}_0 \cdot \nabla)\mathbf{B} \\ &= \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.\end{aligned} \tag{3.7}$$

Thus, we see that Faraday's law is also Galilean invariant[6].

3.2.3. Gauss' law. From equations (2.7), (2.14), (3.4), (3.5) and the formulae in appendix A, recalling $\mathbf{v}_0 = \text{const.}$, we find

$$\nabla' \cdot \mathbf{E}' - \frac{\varrho'}{\varepsilon_0} = \left(\nabla \cdot \mathbf{E} - \frac{\varrho}{\varepsilon_0}\right) - \mathbf{v}_0 \cdot (\nabla \times \mathbf{B}). \tag{3.8}$$

Hence, we see that the Galilean invariance of Gauss' law is assured iff

$$\mathbf{v}_0 \cdot (\nabla \times \mathbf{B}) = 0, \tag{3.9}$$

a condition which is in general not satisfied.

3.2.4. Ampère's law. From equations (2.7), (2.15), (3.4), (3.5) and recalling that Maxwell's constant c—equation (1.1)—is a scalar invariant, we find

$$\begin{aligned}\nabla' \times \mathbf{B}' - \mu_0 \mathbf{j}' - \frac{1}{c^2}\frac{\partial \mathbf{E}'}{\partial t'} &= \left(\nabla \times \mathbf{B} - \mu_0 \mathbf{j} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \\ &\quad + \mu_0 \varrho \mathbf{v_0} - \frac{1}{c^2}(\mathbf{v}_0 \cdot \nabla)\mathbf{E} - \frac{\mathbf{v}_0}{c^2} \times \left[\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right)\mathbf{B}\right],\end{aligned} \tag{3.10}$$

which is null iff

$$\varrho \mathbf{v_0} = \varepsilon_0 \left\{(\mathbf{v}_0 \cdot \nabla)\mathbf{E} + \mathbf{v}_0 \times \left[\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right)\mathbf{B}\right]\right\}; \tag{3.11}$$

again, a condition which is in general not satisfied.

[6] As a matter of fact, this invariance is a *constitutive* property of Faraday's law, being implied by the natural requirement of reciprocity between the two situations: a moving circuit in an external stationary magnetic field and a fixed circuit immersed into a time-varying magnetic field (see [2], section 6.1).

3.3. The charge continuity equation

Thus, our pre-relativistic physicist has discovered that Galilean invariance holds true for the first two Maxwell's equations, while it fails when the second pair of Maxwell's equations is considered. This fact gives rise to a useful pedagogical remark. Students are (usually) well aware of the fact that the electric charge conservation implies a charge continuity equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varrho V) = 0 \quad \Longrightarrow \quad \nabla \cdot \mathbf{j} + \frac{\partial \varrho}{\partial t} = 0, \tag{3.12}$$

as follows from recalling

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \nabla \cdot \mathbf{v}, \tag{3.13}$$

the first formula in appendix A, and definition (2.10). They also generally well know that Maxwell's equations are fully consistent with the charge continuity equation, since it can actually be derived from equations (2.14) and (2.15). On the basis of such a deep interlink between the charge continuity equation and the second pair of Maxwell's equations, students might be tempted to deduce that the lack of Galilean invariance of the latter would naturally 'propagate' to the former as well. In order to avoid this hasty conclusion, students may be invited to put their deduction to the test directly by using Maxwell's equations (2.14) and (2.15) and *not* referring to equations (2.9) and (2.11) at all. From the results in subsections 3.2.3 and 3.2.4 and recalling equations (2.7) and (3.4), they will find

$$\varrho' = \varepsilon_0[\nabla' \cdot \mathbf{E}' + \mathbf{v}_0 \cdot \nabla' \times \mathbf{B}'], \tag{3.14}$$

$$\mathbf{j}' = \frac{1}{\mu_0}\nabla' \times \mathbf{B}' - \varrho \mathbf{v}_0 - \varepsilon_0 \left[\frac{\partial}{\partial t'}(\mathbf{E}' - \mathbf{v}_0 \times \mathbf{B}') - (\mathbf{v}_0 \cdot \nabla)\mathbf{E}\right]. \tag{3.15}$$

Taking the primed divergence of (3.15) and recalling equations (2.7), (2.14), (3.14) and the formulae in appendix A, it therefore follows that

$$\begin{aligned}\nabla' \cdot \mathbf{j}' &= [\varepsilon_0 \nabla \cdot (\mathbf{v}_0 \cdot \nabla)\mathbf{E} - \nabla \cdot (\varrho \mathbf{v}_0)] - \varepsilon_0 \nabla' \cdot \left[\frac{\partial}{\partial t'}(\mathbf{E}' - \mathbf{v}_0 \times \mathbf{B}')\right] \\ &= [(\mathbf{v}_0 \cdot \nabla)(\varepsilon_0 \nabla \cdot \mathbf{E}) - \mathbf{v}_0 \cdot \nabla \varrho] - \frac{\partial}{\partial t'}\{\varepsilon_0[\nabla' \cdot \mathbf{E}' + \mathbf{v}_0 \cdot \nabla' \times \mathbf{B}']\} \\ &= -\frac{\partial \varrho'}{\partial t'}.\end{aligned} \tag{3.16}$$

This shows that the Galilean non-invariance of Maxwell's equations (2.14) and (2.15) actually does *not* affect the charge continuity equation at all. Thus, students will appreciate the fact that an equation obtained from two Galilean non-invariant equations can notwithstanding be Galilean invariant itself—which shows that the invariance issue is not a trivial one indeed.

3.4. What if we neglect the displacement current?

A final curiosity may remain: inclusion of the displacement current term in equation (2.15) was an exceedingly fruitful 'invention' due to Maxwell's genius [1, 2]; it is this very term which allows for the existence of electromagnetic waves and which ultimately guarantees *Lorentz* invariance to Maxwell's theory. Were this term neglected[7], would the above conclusions of our pre-relativistic physicist be altered somehow?

[7] A ucronic divertissement on Maxwell's equations without the Faraday term (equivalent to considering the electric limit) is presented in [20].

Consider the (pre-Maxwell) set of equations:

$$\nabla \cdot \mathbf{B} = 0, \tag{3.17}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.18}$$

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0}, \tag{3.19}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{3.20}$$

First, it can be immediately observed that this set no longer contains the charge continuity equation, but the relation

$$\nabla \cdot \mathbf{j} = 0 \tag{3.21}$$

instead, which is compatible with charge conservation only when

$$\partial \varrho / \partial t = 0. \tag{3.22}$$

Second, the modified Ampère's law (3.20) implies, using equations (2.7) and (3.4), that the current density should behave as

$$\mathbf{j}' = \mathbf{j}, \tag{3.23}$$

which is consistent with (2.11) only when $\varrho \mathbf{v}_0 = 0$, i.e., when $\varrho = 0$ (allowing for the frame change). This requirement lets (3.19) be Galilean invariant only iff

$$\mathbf{v}_0 \cdot \mathbf{j} = 0, \tag{3.24}$$

a condition which is the pre-Maxwell equivalent to (3.9). We therefore see that not even the absence of the displacement term in Ampère's equation can lead to a consistent pre-relativistic Galilean picture of Maxwell electromagnetism[8].

4. Conclusions

In this pedagogical paper we have played the role of a pre-relativistic physicist, showing how the Galilean non-invariance of classical electromagnetism can be proved without recourse to the historically posterior knowledge of the Lorentz transformations. Such a Galilean non-invariance cannot simply be justified 'at a glance'—as students not infrequently do—invoking the two facts: (a) that Maxwell's equations contain a parameter, c, which is dimensionally a velocity and (b) that *any* velocity is non-invariant in Galilean relativity. Indeed, the definition of Maxwell's parameter c in terms of observer-independent quantities implies its scalar invariant character under frame transformations; hence, no clue to the Galilean non-invariance actually comes from the presence of this parameter c in Maxwell's equations.

We have seen that a consistently pre-relativistic set of Galilean transformations for the fields can be obtained, and that Galilean invariance characterizes the first pair of Maxwell's equations, while it fails for the second pair. This lack of invariance does not affect the charge continuity equation, though, even if the latter can actually be derived exactly from the second pair of Maxwell's equations. As a final note, we have also remarked that the loss of Galilean invariance cannot be traced back to the presence of the displacement current term: the lack of Galilean invariance also characterizes the 'pre-Maxwell' theory.

Hence, our pre-relativistic physicist, who has been able to individuate a fully consistent set of Galilean transformation laws for the fields, has finally also been able to prove the Galilean non-invariance of classical electromagnetism, resting on simple pre-relativistic considerations alone.

[8] Note that the symplectic approach presented in [21] deals with the pre-Maxwell equations without charge and current sources (cf equations (44) of [21]) and with different definitions for the fields (cf equations (52) of [21]).

Appendix A. Useful vector identities

Let a be a scalar and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ vectors; the following identities hold:

$$\nabla \cdot (a\mathbf{A}) = \mathbf{A} \cdot \nabla a + a\nabla \cdot \mathbf{A} \tag{A.1}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{A.2}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{A.3}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{A.4}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{A.5}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{A.6}$$

Appendix B. Generalization to arbitrary units

Maxwell's equations in arbitrary units read ([2], appendix)

$$\nabla \cdot \mathbf{B} = 0, \tag{B.1}$$

$$\nabla \times \mathbf{E} = -k_3 \frac{\partial \mathbf{B}}{\partial t}, \tag{B.2}$$

$$\nabla \cdot \mathbf{E} = 4\pi k_1 \varrho, \tag{B.3}$$

$$\nabla \times \mathbf{B} = 4\pi k_2 \alpha \mathbf{j} + \frac{k_2 \alpha}{k_1} \frac{\partial \mathbf{E}}{\partial t}, \tag{B.4}$$

where k_1, k_2, k_3 and α are all universal proportionality *constants* (between the electrostatic force and charges:

$$F_1 = k_1 \frac{qq'}{r^2}; \tag{B.5}$$

magnetic force and currents:

$$\frac{\mathrm{d}F_2}{\mathrm{d}l} = 2k_2 \frac{ii'}{d}; \tag{B.6}$$

electromotive force and varying magnetic flux:

$$\mathcal{E} = -k_3 \frac{\mathrm{d}\Phi}{\mathrm{d}t}; \tag{B.7}$$

and the magnetic induction field and current:

$$B = 2k_2 \alpha \frac{\mathrm{i}}{d}, \tag{B.8}$$

respectively), such that $[k_3] = [\alpha^{-1}]$ and $[k_1/k_2] = \mathrm{L}^2\mathrm{T}^{-2}$, the latter implying that we can define a new *constant*:

$$c \equiv \sqrt{\frac{k_1}{k_2}}, \tag{B.9}$$

with the dimensions of a velocity. The scalar invariant character of c is implied by its very definition; as far as its numerical value is regarded, it can be experimentally determined, cf [3, 15], and it turns out to coincide—within measurement errors—with the independently measured speed of light in vacuo. It is this numerical coincidence which lets one usually speak of the parameter c defined in (B.9) as 'the speed of light in vacuo' *tout court*. Focusing on its being a 'velocity' however also lets one usually forget that it is a 'constant' as well, in pre-relativistic physics too—which forgetfulness naturally induces, as we argued above, the common misunderstanding about the presence of c in Maxwell's equations being the clear sign of their Galilean non-invariance.

Together with the generalized form (B.1)–(B.4) of Maxwell's equations, we also need the generalized form of the Lorentz force law, namely,

$$\mathbf{F} = q(\mathbf{E} + k_{\mathrm{L}}\mathbf{v} \times \mathbf{B}), \tag{B.10}$$

where

$$E = k_1 \frac{q}{r^2} \tag{B.11}$$

is the generalized modulus of the electric field, cf equation (B.5), and k_{L} is yet another proportionality constant (the total thus amounting to five) of the theory. The generalized transformation laws for the fields, following from the request for Lorentz force invariance (cf section 3.1 above), read

$$\mathbf{B}'(\mathbf{x}', t') = \mathbf{B}(\mathbf{x}, t), \tag{B.12}$$

$$\mathbf{E}'(\mathbf{x}', t') = \mathbf{E}(\mathbf{x}, t) + k_{\mathrm{L}}\mathbf{v}_0 \times \mathbf{B}(\mathbf{x}, t). \tag{B.13}$$

While the natural symmetry (see footnote 6) for Faraday's law requires $k_3 = k_{\mathrm{L}}$, coherence of the Lorentz with the Ampère's force law requires $\alpha = k_{\mathrm{L}}^{-1}$. Hence, recalling definition (B.9) and its identification with the measured speed of light in vacuo ($c = 2.998 \times 10^8$ m s^{-1}), we see that out of the five parameters k_1, k_2, k_3, α and k_{L} introduced so far, only two can actually be chosen at will, the remaining three being automatically determined by this choice (for instance, in the SI employed above, one sets $\{k_2 = \mu_0/4\pi \equiv 10^{-7}$ [MLT^{-2} I^{-2}], $k_{\mathrm{L}} \equiv 1\}$, and consequently has $\{k_1 = 1/4\pi\varepsilon_0 = 10^{-7}c^2$ [ML^3 T^{-4} I^{-2}], $k_3 = 1 = \alpha\}$; in the Gaussian system, one takes $\{k_1 \equiv 1, k_{\mathrm{L}} \equiv c^{-1}$ [L^{-1} T]$\}$, and consequently finds $\{k_2 = c^{-2}$ [L^{-2} T^2], $k_3 = c^{-1}$ [L^{-1} T], $\alpha = c$[LT^{-1}]$\}$; and so on, for the other systems of physical units—cf [2], appendix, table 1).

Using equations (B.1)–(B.4) together with (B.12) and (B.13) and the Galilean transformation laws (2.7), (2.9), (2.11), we can again follow the path already traced in subsections (3.2, 3.3) and (3.4) to find that

- the magnetic Gauss' law is Galilean invariant:

$$\nabla' \cdot \mathbf{B}' = \nabla \cdot \mathbf{B} = 0; \tag{B.14}$$

- the Galilean invariance of Faraday's law (see footnote 6) is immediately verified:

$$\begin{aligned} \nabla' \times \mathbf{E}' + k_3 \frac{\partial \mathbf{B}'}{\partial t'} &= \left(\nabla \times \mathbf{E} + k_3 \frac{\partial \mathbf{B}}{\partial t}\right) + k_3\left[\nabla \times (\mathbf{v}_0 \times \mathbf{B}) + (\mathbf{v}_0 \cdot \nabla)\mathbf{B}\right] \\ &= \nabla \times \mathbf{E} + k_3 \frac{\partial \mathbf{B}}{\partial t} = 0; \end{aligned} \tag{B.15}$$

- for Gauss' law, we have

$$\nabla' \cdot \mathbf{E}' - 4\pi k_1 \varrho' = (\nabla \cdot \mathbf{E} - 4\pi k_1 \varrho) - k_3 \mathbf{v}_0 \cdot (\nabla \times \mathbf{B}); \tag{B.16}$$

hence, Galilean invariance would require

$$\mathbf{v_0} \cdot (\nabla \times \mathbf{B}) = 0, \tag{B.17}$$

which is in general not true;

- for Ampère's law, we find

$$\nabla' \times \mathbf{B}' - 4\pi k_2\alpha \mathbf{j}' - \frac{k_2\alpha}{k_1}\frac{\partial \mathbf{E}'}{\partial t'} = \left(\nabla \times \mathbf{B} - 4\pi k_2 \alpha \mathbf{j} - \frac{k_2\alpha}{k_1}\frac{\partial \mathbf{E}}{\partial t}\right) + 4\pi k_2\alpha \varrho \mathbf{v}_0 - \frac{k_2\alpha}{k_1}(\mathbf{v}_0\cdot\nabla)\mathbf{E} - \frac{k_2k_3\alpha}{k_1}\mathbf{v}_0 \times \left[\left(\frac{\partial}{\partial t} + \mathbf{v}_0\cdot\nabla\right)\mathbf{B}\right], \tag{B.18}$$

i.e., Galilean invariance would require

$$\varrho \mathbf{v}_0 = \frac{1}{4\pi k_1}\left\{(\mathbf{v}_0\cdot\nabla)\mathbf{E} + k_3\mathbf{v}_0 \times \left[\left(\frac{\partial}{\partial t} + \mathbf{v}_0\cdot\nabla\right)\mathbf{B}\right]\right\}, \tag{B.19}$$

which is in general not true either;

- the transformation laws for the charge and the charge current density, as directly derived from the second couple of Maxwell's equations (the two Galilean non-invariant ones), namely,

$$\varrho' = \frac{1}{4\pi k_1}[\nabla'\cdot\mathbf{E}' + k_3\mathbf{v}_0\cdot\nabla'\times\mathbf{B}'], \tag{B.20}$$

$$\mathbf{j}' = \frac{1}{4\pi k_2\alpha}\nabla'\times\mathbf{B}' - \varrho\mathbf{v}_0 - \frac{1}{4\pi k_1}\left[\frac{\partial}{\partial t'}(\mathbf{E}' - k_3\mathbf{v}_0\times\mathbf{B}') - (\mathbf{v}_0\cdot\nabla)\mathbf{E}\right], \tag{B.21}$$

do imply the Galilean invariance of the charge continuity equation:

$$\begin{aligned}\nabla'\cdot\mathbf{j}' &= \left[\frac{1}{4\pi k_1}\nabla\cdot(\mathbf{v}_0\cdot\nabla)\mathbf{E} - \nabla\cdot(\varrho\mathbf{v}_0)\right] - \frac{1}{4\pi k_1}\nabla'\cdot\left[\frac{\partial}{\partial t'}(\mathbf{E}' - k_3\mathbf{v}_0\times\mathbf{B}')\right]\\ &= \left[(\mathbf{v}_0\cdot\nabla)\left(\frac{1}{4\pi k_1}\nabla\cdot\mathbf{E}\right) - \mathbf{v}_0\cdot\nabla\varrho\right] - \frac{\partial}{\partial t'}\left\{\frac{1}{4\pi k_1}[\nabla'\cdot\mathbf{E}' + k_3\mathbf{v}_0\cdot\nabla'\times\mathbf{B}']\right\}\\ &= -\frac{\partial\varrho'}{\partial t'};\end{aligned} \tag{B.22}$$

- finally, it can immediately be checked that the 'pre-Maxwell' equations,

$$\nabla\cdot\mathbf{B} = 0, \tag{B.23}$$

$$\nabla\times\mathbf{E} = -k_3\frac{\partial\mathbf{B}}{\partial t}, \tag{B.24}$$

$$\nabla\cdot\mathbf{E} = 4\pi k_1\varrho, \tag{B.25}$$

$$\nabla\times\mathbf{B} = 4\pi k_2\alpha\mathbf{j}, \tag{B.26}$$

do not allow a consistent Galilean picture of classical electromagnetism (equations (3.21)–(3.24) hold unchanged).

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