Physics 7314 — Professor Scalise

Homework Assignment #3

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1. This is problem 4.5 from Griffiths QM book 3rd ed. I think he goofed. At time zero, an electron in a hydrogen atom is put into the state

$$\Phi(\vec{r}) = A\, R_{10}(r)\, Q_{00}(\cos\theta)$$

where $R_{10}(r) = \frac{2}{a_0^{3/2}} \exp(-r/a_0)$ and

$$Q_{00}(\cos\theta) = \frac{1}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) = \ln\left[\cot\left(\frac{\theta}{2}\right)\right] = \operatorname{artanh}(\cos\theta)$$

is an associated Legendre function of the second kind.

(a) Plot $Q_{00}(\cos\theta)$. Griffiths calls this an "unacceptable" solution because the wave function is infinite, but the wave function is not measurable.

(b) In fact, this state can be normalized. What is the modulus of the normalization constant A? Remember to use the correct measure for spherical polar integration.

(c) What is the expectation value of $\hat{L}^2$, $\langle\Phi|\hat{L}^2|\Phi\rangle$, where $\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$?

(d) What is the expectation value of $\hat{L}_z$, $\langle\Phi|\hat{L}_z|\Phi\rangle$, where $\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$?

(e) What is the expectation value of the Hamiltonian, $\langle\Phi|\hat{H}|\Phi\rangle$, where $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}$?

2. For spin angular momentum operators,

(a) what is the commutator $[\hat{S}_x, \hat{S}_y]$?

The state $|+\rangle_z$ means spin up along the z direction. For a spin-$\frac{1}{2}$ particle in the state $\psi = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z)$, what are:

(b) $\langle\hat{S}_z\rangle$, the expectation value of the z-component of spin?

(c) $\langle\hat{S}_x\rangle$, the expectation value of the x-component of spin?

(d) $\langle\hat{S}_y\rangle$, the expectation value of the y-component of spin?

(e) $\Delta S_x = \sigma_{S_x}$, the uncertainty in the x-component of spin?

(f) $\Delta S_y = \sigma_{S_y}$, the uncertainty in the y-component of spin?

(g) $\Delta S_x \Delta S_y$, the Heisenberg product of uncertainties?

(h) Is the Uncertainty Principle violated? Explain.