FROM THE PREVIOUS LECTURE WE KNOW THE FOLLOWING: DATE: 08/29/25

$x^\mu = (ct, \vec{r})$

$x_\mu = g_{\mu\nu} x^\nu = g_{\mu\sigma} x^\sigma$

* WHERE THE SUMMED OVER INDEX IS KNOWN AS THE "DUMMY"

THE METRIC TENSOR, IN MATRIX FORM LOOKS AS FOLLOWS:

$$[g_{\mu\nu}] = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad [g^{\mu\nu}] = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

THE METRIC TENSOR IS CONSTRUCTED SUCH THAT $[g^{\mu\nu}][g_{\mu\nu}] = I_4$, WHERE I_4 IS THE 4×4 IDENTITY MATRIX, THEREFORE $[g^{\mu\nu}] = [g_{\mu\nu}]^{-1}$.

$$= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} = [I_4]$$

MORE PROPERTIES OF THE METRIC TENSOR IN QFT:

$$g_{\mu\nu} g^{\nu\sigma} = \sum_{\nu=0}^{3} g_{\mu\nu} g^{\nu\sigma} = g_\mu{}^\sigma = g^\sigma_\mu = \underline{\delta^\sigma_\mu}$$

* WHERE δ^σ_μ IS THE DELTA FUNCTION

$$\delta^\sigma_\mu = \begin{cases} 1, & \text{if } \mu = \sigma \\ 0, & \text{if } \mu \neq \sigma \end{cases}$$

$$g_{\mu\nu} g^{\mu\nu} = g^\mu_\mu = g^\nu_\nu = \delta^\mu_\mu = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 \underline{= 4}$$

* THIS WILL BE THE NUMBER OF DIMENSIONS OF THE FIELD THEORY

NOTE: $g_{\mu\nu} g^{\mu\nu} \neq tr(g^{\mu\nu})$, WHERE $tr(g^{\mu\nu})$ IS THE TRACE OF THE METRIC TENSOR, THIS IS DUE TO THE RELATIVE SIGN CHANGE IN THE "TIME" AND "SPACE" INDICES OF THE TENSOR.

DERIVATIVES IN QFT ARE OBJECTS THAT ACT ON FIELDS OR 4-VECTORS:

COVARIANT DERIVATIVES $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (\partial_t, \vec{\nabla})$

$$\partial^\mu = {}^{\mu\nu}\partial_\nu = (\partial_t, -\vec{\nabla})$$

$$\partial_\mu x^\nu = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu = g^\nu_\mu$$

$$\partial_\mu x^\mu = \delta^\mu_\mu = 4$$

THESE EXPRESSIONS HAVE ANALOGS IN 3 DIMENSONS AS THE LAPLACIAN OPERATOR:

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \Rightarrow \vec{\nabla}\cdot\vec{r} = \delta^{(3)} = 3$$

$$\Delta = \nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

4 - D'ALEMBERTIAN:

IF IN THREE DIMESIONS THE OPERATOR ($\vec{\nabla}$) IS OBVIOUSLY A TRIANGLE, THEN IT FOLLOWS SUCCINTLY THAT THE FOUR DIMENSONAL OPERATOR SHOULD BE A SQUARE ($\square$)... AND NOT A TETRAHEDRON.

$$\square = \square^2 = \partial_\mu \partial^\mu = \sum_{\mu=0}^{3} \partial_\mu \partial^\mu = \partial_0\partial^0 + \partial_1\partial^1 + \partial_2\partial^2 + \partial_3\partial^3 = \underline{\partial_0^2 - \vec{\nabla}^2}$$

KNOWN AS THE WAVE OPERATOR

$$= \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3 = \partial_0^2 - \underline{\vec{\nabla}\cdot\vec{\nabla}}$$

3 DIMENSONAL DOT PRODUCT

MOVING INTO LORENTZ GROUP AND LORENTZ TRANSFORMATION PROPERTIES:

LORENTZ INVARIANCE IS THE PRINCIPLE THAT FUNDAMENTAL PHYSICAL LAWS ARE THE SAME FOR ALL OBSERVERS IN INERTIAL REFERENCE FRAMES REGARDLESS OF RELATIVE MOTION.

A CHANGE OF FRAME FROM $S \rightarrow S'$ REFERENCE FRAMES (ROTATIONS AND BOOSTS), NOT INCLUDING TRANSLATIONS.

$$x'^{\mu} \equiv \Lambda^{\mu}{}_{\nu}\, x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu}\, x^{\nu} = \underline{\underline{\Lambda}}\,\underline{x} \qquad \text{(EQ 1)}$$

$$y'^{\mu} \equiv \Lambda^{\mu}{}_{\nu}\, y^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu}\, y^{\nu} = \underline{\underline{\Lambda}}\,\underline{y}$$

$$y'_{\mu} = g_{\mu\sigma}\Lambda^{\sigma}{}_{\nu}\, y^{\nu} = \sum_{\sigma=0}^{3}\sum_{\nu=0}^{3} g_{\mu\sigma}\Lambda^{\sigma}{}_{\nu}\, y^{\nu}$$

NOTE THAT Λ HAS THE FOLLOWING PROPERTIES:

$$\Lambda^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \Rightarrow (\Lambda^{-1})^{\mu}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}$$

IN 4 DIMESIONS WE CAN FIND THE FOLLOWING RELATIONSHIP:

KNOWING THAT THE DOT PRODUCT IS INVARIANT UNDER LORENTZ TRANSFORMATIONS

$$x' \cdot y' = x'^{\mu} y'_{\mu} = x^{\mu} y_{\mu} = x \cdot y$$

FROM (EQ 1) $\quad x'_{\mu} = \left[(\Lambda^{-1})^{T}\right]_{\mu}{}^{\nu} x_{\nu} = (\Lambda^{-1})^{\nu}{}_{\mu}\, x_{\nu} = x_{\nu}(\Lambda^{-1})^{\nu}{}_{\mu}$

$$= \underline{x}\,\underline{\underline{\Lambda}}^{-1}$$

THE RESULT WE EXPECT FROM VERIFYING THIS PROPERTY IS THE IDENTITY MATRIX

$x' \cdot y' = x \cdot y$

$$\underbrace{\Lambda^{\mu}{}_{\nu}\, x^{\nu}}_{x'}\; g_{\mu\rho}\; \underbrace{\Lambda^{\rho}{}_{\sigma}\, y^{\sigma}}_{y'} = x^{\nu} g_{\nu\sigma}\, y^{\sigma} \Rightarrow$$

$$\underbrace{\Lambda^{\mu}{}_{\nu}\, g_{\mu\rho}\, \Lambda^{\rho}{}_{\sigma} = g_{\nu\sigma}}_{\underline{\underline{\Lambda}}^{T}\underline{\underline{g}}\,\underline{\underline{\Lambda}} = \underline{\underline{g}}} \Rightarrow \underbrace{(\Lambda^{T})_{\nu}{}^{\mu}\, g_{\mu\rho}\, \Lambda^{\rho}{}_{\sigma} = g_{\nu\sigma}}_{\underline{\underline{g}}^{-1}\underline{\underline{\Lambda}}^{T}\underline{\underline{g}}\,\underline{\underline{\Lambda}} = \underline{\underline{I}}_{4}}$$

RECALL IN 3 DIMESONS ROTATIONS ARE COMPUTED AS FOLLOWS :

$\vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y}$ * THE DOT PRODUCT BETWEEN TWO VECTORS WILL BE SAME IN THE S' AND S FRAMES, THIS IS LORENTZ INVARIANCE.

$$\vec{x}' = \underline{\underline{R}}\,\vec{x} \ , \ \vec{y}' = \underline{\underline{R}}\,\vec{y} \Rightarrow x_i' = \sum_{j=1}^{3} R_{ij}\, x_j \ , \ y_i' = \sum_{j=1}^{3} R_{ik}\, y_k$$

$$\sum_{j=1}^{3} \underline{R_{ij}\, x_j\, R_{ik}\, y_k} = x_j\, \delta_{jk}\, y_k \qquad \sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk}$$

ONE FINAL NOTE ABOUT THE PROPERTIES OF THE ROTATION MATRICES

$$\sum_{i=1}^{3} (R^T)_{ji}\, R_{ik} = \delta_{jk} \Rightarrow \underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{I}}_3$$