REVIEW OF QM CONTINUED

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$$|a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \qquad |b\rangle = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}$$

YOU CAN CONSTRUCT THE INNER PRODUCT OR SCALAR PRODUCT

$$\langle a|b\rangle = c$$

* KET VECTOR LIVES IN A HILBERT SPACE $\mathcal{H}$ (POSSIBLY ∞ DIMENSIONAL)
* BRA COVECTOR LIVES IN A DIFFERENT HILBERT SPACE $\mathcal{H}^*$ AS THE DUAL OF $\mathcal{H}$
* A COMPLEX NUMBER (C-NUMBER) LIVES IN A ONE DIMENSIONAL HILBERT SPACE

$$\langle a|b\rangle = \underbrace{[a_1^*, a_2^*, a_3^*, \cdots]}_{1\times N} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}}_{N\times 1} = \underbrace{c}_{1\times 1} = (a_1^*)b_1 + (a_2^*)b_2 + (a_3^*)b_3 + \cdots$$

THE OUTER PRODUCT PRODUCES OPERATORS AS FOLLOWS

$$|b\rangle\langle a| = \text{OPERATOR} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}}_{N\times 1} \underbrace{[a_1^*, a_2^*, \cdots]}_{1\times N} = \underbrace{\begin{bmatrix} b_1 a_1^* & b_1 a_2^* & b_1 a_3^* & \cdots \\ b_2 a_1^* & b_2 a_2^* & b_2 a_3^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{N\times N}$$

WHAT CAN ACT ON THE KETS $|\phi\rangle$?

NOT LORENTZ MATRICIES $(\Lambda^\mu{}_\nu)$ LIKE $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\nu$

OPERATORS CAN ACT ON KETS $|\phi\rangle$ IN HILBERT SPACES LIKE $\hat{x}$, $\hat{P}_x$, $\hat{H}$. OPERATORS GET HAVE A PARTICULAR NOTATION, THEY RECIEVE HATS (^) LIKE $\hat{P}_x$ OR $\hat{\vec{P}}$.

WIGNER'S THEOREM: BROADLY, THE INTRINSIC NOTION OF SYMMETRIES IN QUANTUM PHYSICS ASSUMING ONLY OBSERVEABLE PROPERTIES OF QUANTUM STATES AND DERIVING FROM THIS QUANTUM SYMMETRIES ARE NECESSARILY REPRESENTED BY UNITARY OR ANTI-UNITARY OPERATORS ON THE HILBERT SPACE OF STATES.

UNITARY OPERATORS $\hat{U}$, $\langle \hat{U}\psi | \hat{U}\phi \rangle = \langle \psi | \hat{U}\hat{U}^{\dagger} | \phi \rangle = \langle \psi | \phi \rangle$

ANTIUNITARY OPERATORS $\hat{A}$, $\langle \hat{A}\psi | \hat{A}\phi \rangle = \langle \phi | \psi \rangle = \langle \psi | \phi \rangle^{*}$

FOCUSING ON UNITARITY OPERATORS FOR NOW WE CAN DERIVE THE FOLLOWING:

$\underline{\hat{U}\hat{U}^{\dagger} = \hat{1}}$ $\Rightarrow$ $\hat{U}^{\dagger} = \hat{U}^{-1}$ DEPENDS ON SOME CONTINOUS PARAMETER(S)

* UNITARITY OPERATORS PRESERVE THE NORM

- $\hat{U}(d)\,\hat{U}^{\dagger}(e) = \hat{1}$ INVERSE
- $\hat{U}(d)\,\hat{U}(e) = \hat{U}(d \circ e)$ COMPOSITION
- $\hat{U}(0) = \hat{1}$ IDENTITY

THESE TRANSFORMATIONS, $\hat{U}$ FORM A GROUP, A LIE GROUP.

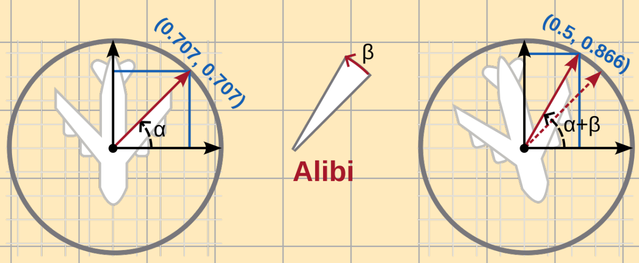
THE FIRST EXAMPLE OF UNITARY OPERATORS, THE SPATIAL TRANSFORMATION:

$$\underline{\hat{U}(a)} = \exp\left\{-\frac{i}{\hbar}\hat{p}\,a\right\} \xrightarrow{\hbar = 1} = \exp\left\{-i\hat{p}\,a\right\}$$

* 1 DIMENSIONAL, THE "x" DISTANCE

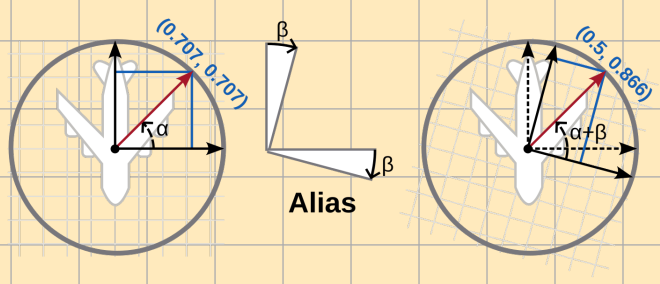
$\hat{p}$ IS THE HERMITIAN MOMENTUM OPERATOR $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ IN COORDINATE REPERSENTATION. (CARTESIAN)

ACTIVE VS PASSIVE TRANSFORMATIONS



ACTIVE TRANSFORMATIONS : CHANGE THE PHYSICAL POSITION OF A SET OF POINTS RELATIVE TO A FIXED FRAME OF REFERENCE

"ALIBI TRANSFORMATION"



PASSIVE TRANSFORMATIONS : LEAVE POINTS FIXED BUT CHANGE THE FRAME OF REFERENCE TO WHICH THEY ARE DESCRIBED.

"ALIAS TRANSFORMATIONS"

TAYLOR EXPAND THE WAVE FUNCTION TO SEE TRANSLATIONS

$$\psi(x+\delta a) = \psi(x) + \frac{d\psi(x)}{dx}\delta a + \cdots$$

$$= (\hat{\mathbb{1}} + i\hat{p}\,\delta a)\,\psi(x)$$

$$\psi(x+a) = \lim_{N\to\infty}\left(\hat{\mathbb{1}} + i\hat{p}\,\delta a\right)^N \psi(x) = e^{i\hat{p}a}\,\psi(x)$$

SPACE EVOLUTION - PASSIVE TRANSFORMATION

SPACE TRANSFORMATION - ACTIVE TRANSFORMATIONS (SIGN DIFFERENCE)

3 DIMENSONAL SPATIAL TRANSFORMATIONS

$$\hat{U}(\vec{a}) = \exp\{-i\,\hat{\vec{p}}\cdot\vec{a}\}$$

TIME TRANSLATIONS

$$\hat{U}(t_a) = \exp\{+i\,\underline{\hat{H}}\,t_a\}$$

* $\hat{H}$ IS THE HERMITIAN HAMILTONIAN ENERGY OPERATOR

FOUR DIMENSONAL SPACE-TIME TRANSFORMATIONS

$$\hat{U}(\underline{a}) = \exp\{+i\hat{H}t_a - i\,\hat{\vec{p}}\cdot\vec{a}\} = \exp\{+i\,\hat{p}^{\mu}a_{\mu}\} = \exp\{+\underline{i\,\hat{p}\cdot a}\}$$

* FOUR DIM. DOT PRODUCT