SPATIAL TRANSLATIONS

DATE: 09/12/2025

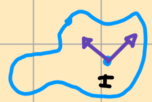
$$\hat{U}(\vec{a}) = \exp\{-i\,\hat{\vec{P}}\cdot\vec{a}\}$$

* LIE GROUP ELEMENT

* GENERATOR OF THE LIE ALGEBRA

$$\hat{P}_K = i\,\frac{\partial \hat{U}(\vec{a})}{\partial a_K}\bigg|_{\vec{a}=0}$$

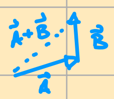
NOTE: A GROUP ELEMENT IN A GIVEN MANIFOLD, HAS AN ASSOCIATED LIE GROUP ELEMENT SUCH THAT $\hat{I} = \hat{U}(0)$



GROUP AXIOM COMPOSITIONS

$$g_1 \circ g_2 = g_3 \Rightarrow \hat{U}(\vec{b})\,\hat{U}(\vec{a}) = \hat{U}(\vec{a}+\vec{b})$$

* THIS CAN BE VISUALIZED WITH VECTORS



FOR LIE ALGEBRAS OF MOMENTUM HAS A SIMPLE 4R, REPERSENTATION

$$[\hat{P}_i, \hat{P}_j] = 0 \equiv \hat{P}_i\hat{P}_j - \hat{P}_j\hat{P}_i$$

SPACE-TIME TRANSLATIONS

$$[\hat{P}_\mu, \hat{P}_\nu] = 0$$

EXPECTATION VALUES OF OPERATORS

$$\hat{U}(\vec{a})|0\rangle = |\vec{a}\rangle$$

$$\langle 0|\hat{O}(\vec{x})|0\rangle = \underbrace{\langle 0|\hat{U}^{+}(\vec{a})}_{\langle\vec{a}|}\underbrace{\hat{U}(\vec{a})\hat{O}(x)\hat{U}^{+}(\vec{a})}_{\hat{O}(\vec{x}+\vec{a})}\underbrace{\hat{U}(\vec{a})|0\rangle}_{|\vec{a}\rangle} = \langle\vec{a}|\,\hat{O}(\vec{x}+\vec{a})\,|\vec{a}\rangle$$

$$\hat{O}(\vec{x}+\vec{a}) = \hat{U}(\vec{a})\,\hat{O}(\vec{x})\,\hat{U}^{+}(\vec{a}) = \exp\{-i\,\hat{\vec{P}}\cdot\vec{a}\}\;\hat{O}(\vec{x})\;\exp\{+i\,\hat{\vec{P}}\cdot\vec{a}\}$$

MOMENTUM IS A HERMITIAN OPERATOR, $\hat{\vec{P}}^{+} = \hat{\vec{P}}$ AND FOLLOWS:

"UNITARY" = exp{ i "HERMITIAN" }

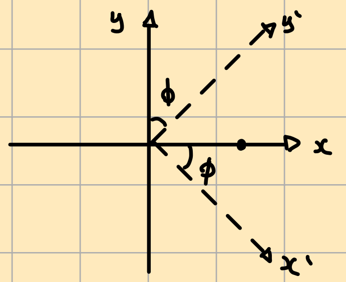
ROTATIONS, ROTATING BY ANGLE θ AROUND z-AXIS IN $SO(3)$:

$$\psi(\phi + \delta\theta) = \psi(\phi) + \frac{d\psi(\phi)}{d\phi}\,\partial\theta + \cdots = (\hat{\mathbb{1}} + i\hat{J}_z\,\partial\theta)\,\psi(\phi)$$

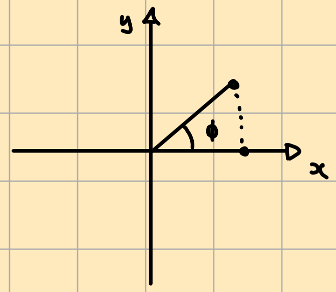
* RECALL FROM QM $\hat{J}_z = -i\frac{\partial}{\partial\phi}$

$$\psi(\phi+\theta) = \lim_{N\to\infty} (\hat{\mathbb{1}} + i\hat{J}_z\,\partial\theta)^N \psi(\phi) = \exp\{i\hat{J}_z\theta\}\,\psi(\phi)$$

GOING FROM PASSIVE TO ACTIVE OR VICE VERSA, GAINS A RELATIVE MINUS SIGN.



(PASSIVE)



(ACTIVE)

$$\hat{U}(\theta) = \exp\{-i\hat{J}_z\theta\}$$

MOVING ON FROM THIS SPECIAL ROTATION TO GENERAL ROTATIONS WE FIND

$$\hat{U}(\vec{\theta}) = \exp\{-i\,\hat{\vec{J}}\cdot\vec{\theta}\}$$

* $\hat{\vec{J}}$: IS THE ANGULAR MOMENTUM OPERATOR
* $\vec{\theta}$ = $\hat{\theta}$ IS THE DIRECTION OF THE AXIS OF ROTATION; θ IS THE ANGLE THROUGH WHICH YOU ROTATE

REPERSENTATIONS

LOOKING AT REPERSENTATIONS OF $\underline{\underline{D}}$ A GROUP OF SQUARE MATRICIES.

IF $g_1 \circ g_2 = g_3$ THEN $\underline{\underline{D}}(g_1)\,\underline{\underline{D}}(g_2) = \underline{\underline{D}}(g_3)$

* $SO(2)$, $U(1)$, $SO(3)$, $SU(2)$

① SCALAR FIELD. EVERY GROUP HAS A TRIVIAL REPERSENTATION ALL g's ARE 1

$$D(\theta) = 1 \Rightarrow \hat{J}_z = i\left.\frac{\partial D(\theta)}{\partial\theta}\right|_{\theta=0} = 0 \text{ , ALSO FOR } \hat{J}_x = \hat{J}_y = 0$$

② SPIN-½ A SPINOR

$$D(\theta) = \begin{bmatrix} \exp\{-\frac{1}{2}i\theta\} & 0 \\ 0 & \exp\{\frac{1}{2}i\theta\} \end{bmatrix} \Rightarrow \hat{J}_z = i\left.\frac{\partial D(\theta)}{\partial\theta}\right|_{\theta=0} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{J}_z = \frac{\sigma_z}{2}$$

* σ_z IS THE PAULI MATRIX

③ SPIN - 1 A VECTOR

$$D(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad J_z = i \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\Bigg|_{\theta=0}$$

$$J_z = i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

THE LIE ALGEBRA FOR VECTOR IS GIVEN BY THE FOLLOWING RELATIONSHIP:

$$\left[\hat{J}_i , \hat{J}_j \right] = i \sum_{k=1}^{3} \underline{\varepsilon_{ijk}} \, \hat{J}_k$$

ε_{ijk} IS THE LEVI-CIVITA SYMBOL IS DEFINED BY:

$$\varepsilon_{ijk} = \begin{cases} +1 , & \text{IF } (i,j,k) \text{ IS } (1,2,3), (2,3,1) \text{ OR } (3,1,2) \\ -1 , & \text{IF } (i,j,k) \text{ IS } (3,2,1), (1,3,2) \text{ OR } (2,1,3) \\ 0 , & \text{IF } i=j \text{ OR } j=k \text{ OR } k=i \end{cases}$$