REVIEW FROM PREVIOUS LECTURE DATE: 09/15/25

TRANSLATIONS $\hat{U}(\vec{a}) = \exp\{-i\,\hat{\vec{p}}\cdot\vec{a}\}$ * WHERE $\hat{U}$ IS A GROUP ELEMENT

$\hat{U}(\vec{b})\,\hat{U}(\vec{a}) = \hat{U}(\vec{a})\,\hat{U}(\vec{b}) = \hat{U}(\vec{a}+\vec{b}) \Rightarrow$ GROUP ELEMENTS COMMUTE

$[\hat{p}_i, \hat{p}_j] = 0 \Rightarrow$ ABELIAN LIE ALGEBRA 3D.

DEFINITION: ABELIAN GROUPS, OR COMMUTATIVE GROUPS ARE GROUPS WHERE THE RESULT OF APPLYING A GROUP OPERATION TO TWO ELEMENTS DO NOT DEPEND ON THE ORDER IN WHICH THEY ARE WRITTEN. THIS WILL BE HELPFUL LATER FOR UNDERSTANDING GAUGE THEORIES.

ROTATIONS $\hat{U}(\vec{\theta}) = \exp\{-i\,\hat{\vec{J}}\cdot\vec{\theta}\}$

$$\hat{U}(\vec{\theta}_2)\,\hat{U}(\vec{\theta}_1) \neq \hat{U}(\vec{\theta}_1)\,\hat{U}(\vec{\theta}_2) \Rightarrow [\hat{J}_i, \hat{J}_j] = i\sum_{k=1}^{3} \varepsilon_{ijk}\,\hat{J}_k$$

LIE ALGEBRAS HAVE THE FOLLOWING GENERATOR:

$$[\hat{T}_a, \hat{T}_b] = \sum_c f_{abc}\,\hat{T}_c = \hat{T}_a\hat{T}_b - \hat{T}_b\hat{T}_a$$

* f_{abc} ARE STRUCTURE CONSTANTS

STRUCTURE CONSTANTS f_{abc}

$SU(2)$ WILL PROVIDE A SIMPLE EXAMPLE OF WHAT STRUCTURE FUNCTIONS ARE, WE KNOW THE FOLLOWING RELATIONSHIP WITH THE PAULI MATRICIES (σ_i):

$$[\sigma_a, \sigma_b] = 2i\,\varepsilon^{abc}\,\sigma_c \Rightarrow f^{abc} = 2i\,\varepsilon^{abc}$$

A LIE GROUP IS COMPACT IF THE PARAMETERS ARE BOUNDED. THEREFORE f_{abc} IS TOTALLY ANTI-SYMMETRIC. $SO(2)$ IS A COMPACT GROUP, WHEREAS THE LORENTZ GROUP $SO(3,1)$ IS NOT COMPACT.

DEFINITION: COMPACT GROUPS ARE A TOPOLOGICAL GROUP WHERE THE UNDERLYING TOPOLOGICAL SPACE IS ITSELF COMPACT SPACE. THE GROUP'S TOPOLOGY SATISFIES THE CONDITION THAT EVERY OPEN COVER OF THE GROUP HAS A FINITE SUBCOVER.

EXAMPLE) BOOSTS IN THE LORENTZ GROUP BY ϕ RAPIDITY ALONG THE x-AXIS

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\hat{U}(\vec{\phi}) = \exp\{+i\, \hat{\vec{K}} \cdot \vec{\phi}\} \quad \Rightarrow \quad \hat{U}(\phi\hat{x}) = \exp\{+i\, \hat{K}_x \phi\}$$

$$\hat{K}_1 = \hat{K}_x = -i \left. \frac{\partial \hat{U}(\phi)}{\partial \phi} \right|_{\phi=0}$$

$$\hat{K}_1 = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{J}_1 = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{bmatrix}$$

$$\hat{K}_2 = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{J}_2 = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\hat{K}_3 = -i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{J}_3 = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

FOR THE $SO(3,1)$ LIE ALGEBRAS HAVE THE FOLLOWING COMMUTATION RELATIONS

$$\left.\begin{aligned} [\hat{J}_i, \hat{J}_j] &= i \varepsilon_{ijk} \hat{J}_k \\ [\hat{K}_i, \hat{K}_j] &= -i \varepsilon_{ijk} \hat{L}_k \\ [\hat{J}_i, \hat{K}_j] &= i \varepsilon_{ijk} \hat{K} \end{aligned}\right\} \quad \hat{U}(\Lambda^\mu{}_\nu) = \exp\left\{-i\left(\hat{\vec{J}}\cdot\vec{\theta} - \hat{\vec{K}}\cdot\vec{\phi}\right)\right\}$$

AN ASIDE, A MATH TRICK TO THINK ABOUT LIE ALGEBRAS:

DEFINE: $\hat{A}_k = \frac{1}{2}(\hat{J}_k + i\hat{K}_k)$ $\qquad$ $\hat{B}_k = \frac{1}{2}(\hat{J}_k - i\hat{K}_k)$

$$\left.\begin{aligned} [\hat{A}_i, \hat{A}_j] &= i \varepsilon_{ijk} \hat{A}_k \\ [\hat{B}_i, \hat{B}_j] &= i \varepsilon_{ijk} \hat{B}_k \\ [\hat{A}_i, \hat{B}_j] &= 0 \end{aligned}\right\} \quad SO(3) \text{ OR } SU(2) \text{ ROTATION}$$

$SO(3,1) \simeq SU(2) \otimes SU(2)$ HAVE THE SAME COMPLEXIFICATION $SL(2,C) \otimes SL(2,C)$

SCHRÖDINGER VERSUS HEISENBERG PICTURE

SCHRÖDINGER PICTURE IS TYPICALLY MORE USEFUL IN QUANTUM MECHANICS. WHEREAS THE HEISENBERG PICTURE IS FAR MORE USEFUL IN QFT.

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle$$

$$|\psi_S(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$$

* SCHRÖDINGER KET IS TIME DEPENDENT

EXPECTATION VALUE OF AN OPERATOR $\hat{O}$:

$$\langle \hat{O} \rangle_t = \langle \psi(t) | \hat{O} | \psi(t) \rangle = \langle \psi(0) | \hat{U}^\dagger(t,0)\, \hat{O}\, \hat{U}(t,0) | \psi(0) \rangle$$