CONTINUING THE PROBLEM FROM THE PREVIOUS LECTURE

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FINDING THE AMPLITUDE FOR THE FREE PARTICLE TO TRAVEL FASTER THAN LIGHT TO "ESCAPE" ITS LIGHT CONE.

$$A = \langle \vec{r} | e^{-i\hat{H}t} | \vec{0} \rangle$$

WE WILL NEED THE FOLLOWING RELATIONSHIP FROM QUANTUM MECHANICS:

$$\langle \vec{r} | \vec{p} \rangle = \frac{1}{(\sqrt{2\pi\hbar})^3} \exp\{ i \vec{p}\cdot\vec{r} / \hbar \}$$

* THIS NORMALIZATION WILL CHANGE IN QFT AND $\hbar = 1$.

$$= \frac{1}{(2\pi)^{3/2}} \exp\{ i \vec{p}\cdot\vec{r} \}$$

IN THE MOMENTUM BASIS WE FIND THE FOLLOWING IDENTITY:

$$\hat{\mathbb{1}} = \int d^3p \, |\vec{p}\rangle\langle\vec{p}|$$

* $dp_x \, dp_y \, dp_z$ OR MORE CONVENIENTLY $p^2 \, dp \, \sin\theta \, dp_\theta \, dp_\phi$

$$= \int dp_x \, dp_y \, dp_z \, |\vec{p}\rangle\langle\vec{p}| = \int p^2 \, dp \, \sin\theta \, dp_\theta \, dp_\phi \, |\vec{p}\rangle\langle\vec{p}|$$

THE KETS WHEN APPLYING THE TIME EVOLUTION OPERATOR IN MOMENTUM SPACE

$$e^{-i\hat{H}t} |\vec{p}\rangle = e^{-iE_p t} |\vec{p}\rangle$$

* $E_p = \sqrt{p^2 + m^2}$

NOW TO FIND THE AMPLITUDE WE FIND:

$$A = \int d^3p \, \langle \vec{r} | e^{-i\hat{H}t} | \vec{0} \rangle \langle \vec{p} | \vec{0} \rangle$$

$$= \int d^3p \, \langle \vec{r} | \vec{p} \rangle \, e^{-iE_p t} \, \frac{1}{(2\pi)^{3/2}} \underbrace{e^{-i\vec{p}\cdot\vec{0}}}_{= e^0 = 1}$$

$$= \int \frac{d^3p}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{r}} \cdot e^{-iE_p t}$$

* TO EVALUATE THIS INTEGRAL WE WILL NEED SPHERICAL POLAR COORDINATES

$$= \int \frac{d^3p}{(2\pi)^3} \, e^{i(\vec{p}\cdot\vec{r} - E_p t)}$$

$$A = \frac{1}{(2\pi)^3} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{=2\pi} \int_{p=0}^{\infty} p^2 dp \underline{\int_{\theta=0}^{\pi} \sin\theta \, d\theta \, e^{ipr\cos\theta}} \, e^{-iE_p t}$$

* Let $\mu = \cos\theta$

$\therefore \int_{\cos\theta=-1}^{+1} d(\cos\theta)$

$$A = \frac{1}{(2\pi)^3} \cdot 2\pi \int_{p=0}^{\infty} p^2 dp \underline{\int_{\mu=-1}^{+1} d\mu \, e^{ipr\mu}} \cdot e^{-iE_p t}$$

$$= \frac{e^{ipr} - e^{-ipr}}{ipr}$$

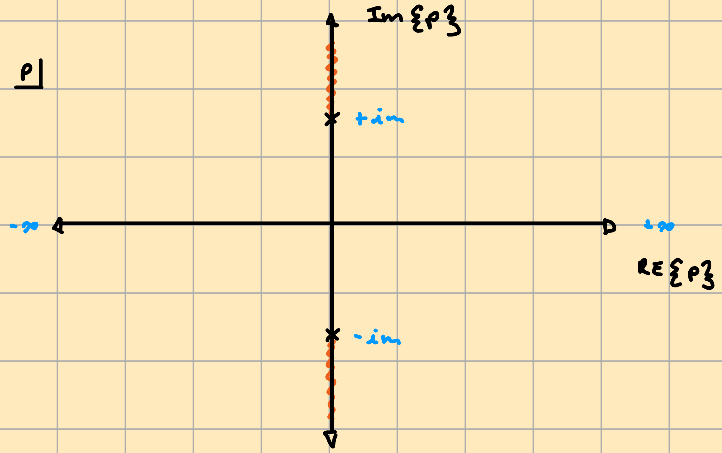
$$A = \frac{1}{(2\pi)^3} \frac{1}{ir} \int_{p=0}^{\infty} dp \, p \left(e^{ipr} - e^{-ipr}\right) e^{-it\sqrt{p^2+m^2}}$$

* Let: $q = -p$

$dq = -dp$

$$A = \frac{-i}{(2\pi)^3 r} \underline{\int_{-\infty}^{\infty} p \, dp \, e^{ipr} e^{-it\sqrt{p^2+m^2}}}$$

* THIS INTEGRAL REQUIRES INTEGRATION IN THE COMPLEX PLANE



$p^2 = -m^2 \Rightarrow$

$p = \pm im$

$\pm im$ ARE BRANCH POINTS THAT ARE CONNECTED WITH A BRANCH CUT IN THE COMPLEX PLANE.

DEFINITION: A BRANCH POINT IS A POINT WHERE A MULTI-VALUED OPERATION GIVES AN AMBIGOUS ANSWER, WITH DIFFERENT BRANCHES GIVING THE SAME OUTPUT.

DEFINITION: A BRANCH CUT IS A CURVE IN THE COMPLEX PLANE THAT IS INTRODUCED TO RESOLVE THE MULTI-VALUED NATURE OF COMPLEX FUNCTIONS, MAKING IT SINGLE VALUED, BY PREVENTING THE FUNCTION FROM COMPLETING A FULL LOOP AROUND A BRANCH POINT.