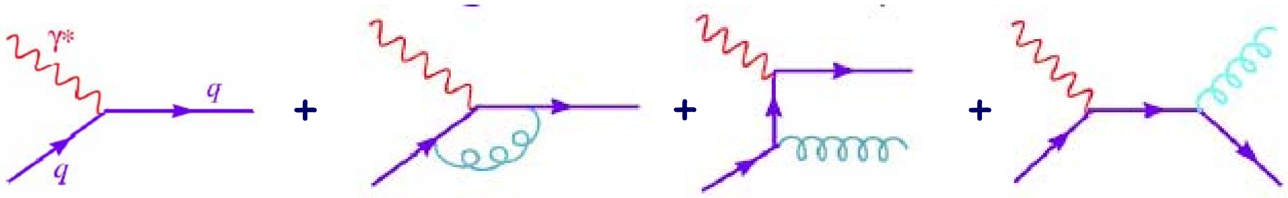
Part IV: General Formalism of PQCD

- Factorization Theorem to all orders of the perturbative expansion (with non-zero quark masses);
- The importance of Scales, Factorization and Renormalization;
- General definition of Parton Distribution Functions (PDF);
- The Three Faces of the Magical Factorization Master Formula;
- Scale dependence of PDFs and QCD Evolution;
- Scale dependence (and independence) of Physical Predictions.

What happens when we consider the other NLO sub-process $\gamma q \to gq$?

Square:



Compared to the $\gamma^{*}g$ case:

- Order α_s terms are divergent: needs UV renormalization;
- Combine real and virtual diagrams to ensure cancellation of infra-red divergences (cf. e^+e^- case);
- Remaining infra-red unsafe terms are associated with long-distance physics: they are universal, and factorizable into a parton distribution function (—in this case, quark/quark).

The IRS hard scattering cross section is the remainder, after the long-distance piece is subtracted (at the factorization scale μ_f).

What Happens at Higher Orders in α_s and for Other Hard Processes?

The same pattern of behavior is repeated order-by-order for most hard processes which has been calculated so far, to 1- or 2-loop level.

To prove the validity of the factorization formula *to all orders of PQCD* is a highly theoretical and technical matter (very sophisticated mathematical induction methods).

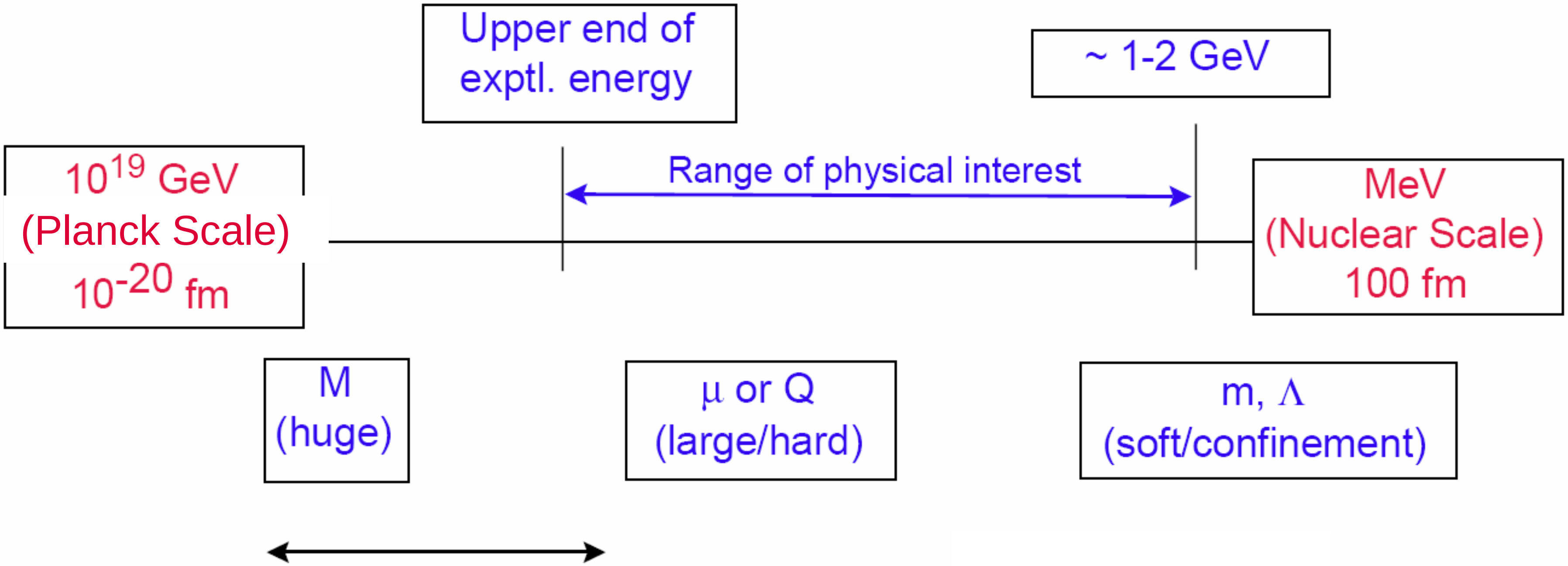
Serious proofs exist only for a limited number of processes such as e^+e^-, DIS and DY. Faith in general factorization rests on existing concrete 1- or 2-loop calculations, and on the suggestive space-time picture.

Traditional proofs of factorization assume m=0 as the starting point. J. Collins showed that factorization in DIS (hence probably other processes as well) is independent of the value of quark masses. (hep-ph/9806259) Our sample calculation with finite quark mass illustrates why this is the case. (The hard cross sections contain IRS mass effects.)

What does a Factorization Thm Proof Consist of?

- It involves the demonstration that, to all orders of α_s in PQCD, a set of *universal parton distribution functions* exists, which will absorb all the colinear and soft singularities encountered in the calculation of all relevant partonic scattering processes.
- Two points should immediately come to mind:
  - This situation is very similar to renormalization, where all (UV) singularities are shown to be absorbed into a set of finite number of renormalization constants; (next two slides)
  - If such universal functions (PDF) exist, there must be a precise definition of them – in a *process-independent* way. (topic after that)

The importance of ***Scales*** -- Renormalization and Factorization



Renormalization Group Equations (RGE) relates physics at different scales

Ultra-violet Renormalization hides / summarizes our ignorance of physics at huge scale in $\alpha_s(\mu_R)$, $m_i(\mu_R)$, …

"Renormalization" and "Factorization"

	UV renormalization		Collinear/soft factorization	
A:	Bare Green Func.	$G_0(\alpha_0, m_0, ..)$	Partonic X-sect	F_a
B:	Ren. constants	$Z_i(\mu)$	Pert. parton dist.	$f_a^b(\mu)$
C:	Ren. Green Fun.	$G_R = G_0/Z$	Hard X-sect	$\hat{F} = F\,/\,f$
D:	Anomalous dim.	$\gamma = \frac{\mu}{Z}\frac{d}{d\mu}Z$	Splitting fun.	$P = \frac{\mu}{f}\frac{d}{d\mu}f$
E:	Phys. para. α, m	$\alpha_0 Z_i \ldots$	Had. parton dist. f_A	resummed
F:	Phys sc. amp.	$\alpha(\mu)\,G_R(m,\mu)$	Hadronic S.F.'s F_A	$f_A(\mu) \times \hat{F}(\mu)$

$$F = f \otimes \hat{F}$$

Some common features:

A : divergent; but, independent of "scheme" and scale μ;

B : divergent; scale and scheme dependent;
universal; absorbs all ultra-violet/soft/collinear divergences;

C & D : finite; scheme-dependent;
D controls the μ dependence of E & F;

E : physical parameters to be obtained from experiment;

F : Theoretical "prediction"; μ-indep. to all orders,
but μ-dep. at finite order n; $\mu\frac{d}{d\mu} \sim \mathcal{O}(\alpha^{n+1})$

Note: "Renormalization" is factorization (of UV divergences);
"factorization" is renormalization (of soft/collinear div.)

$\overline{\text{MS}}$ definition of parton distribution functions

Factorization Theorem requires a precise definition of the parton distribution functions (pdf's).

Note: $F_A = f_A^a \otimes \hat{F}_a \quad \Longrightarrow \quad \hat{F}_a = \frac{F_A}{f_A^a}$

$\Longrightarrow$ the definition of hard scattering cross-sections is determined by that of pdf's.

Historical definition of pdf's: use "operator product expansion" – towers of local operators.

Equivalent modern definition:
use bi-local operators on the light cone:

Definition of PDF's

The distribution of quark "a" in a parent "A" (either hadron or another parton) is:

$$f_A^a(\xi) \equiv \frac{1}{2}\int \frac{dy^-}{2\pi}\, e^{-i\xi p^+ y^-} \langle A|\bar{\psi}_a(0,y^-,\mathbf{0})\gamma^+ F \psi_a(0)|A\rangle.$$

$|A\rangle$ represents the parent state A, with momentum p^μ aligned so that $p_T = 0$ and $p_a^+/p_A^+ = \xi$. This is the matrix element of the appropriate number operator for finding "a" in "A".

The operator

$$F = \mathcal{P}\exp\left(-ig\int_0^{y^-} dz^- A_a^+(0,z^-,\mathbf{0})\, t_a\right)$$

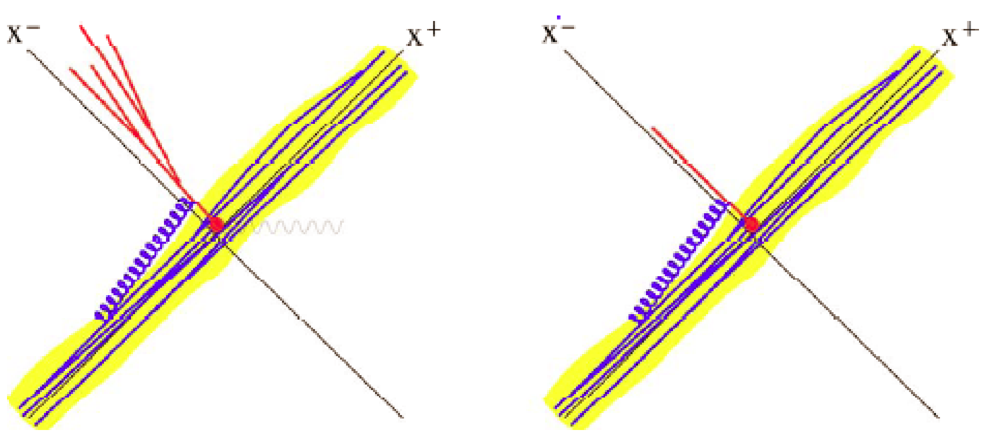
ensures the definition gauge invariant.

There is a similar definition of the gluon distribution inside the hadron A, using the gluon field operator.

- The $\overline{\text{MS}}$ definition in terms of operators is process independent, re-affirming that PDF's are *universal*.
- Sum rules are automatic. Eg.

$$\sum_a \int_0^1 d\xi \, \xi \, f_A^a(\xi, \mu) = 1.$$

Space-time picture of the definition of PDF's is very similar to that of DIS struction functions:
$\langle p|\psi(x)\bar{\psi}(0)|p\rangle$ vs. $\langle p|J(x)J^{\dagger}(0)|p\rangle$:



What does factorization of partonic cross sections have to do with hadronic cross sections.?

From the fact that

- the factorization theorem is proven for partonic cross-sections to all orders in PQCD;
- the parton distributions are universal; and are defined for hadron as well as parton parents.
- the hard cross-section is infra-red safe;

it is natural to assume,

Hadronic cross-sections, such as $F_A^\lambda(x,Q)$, satisfy the same factorization formula as the partonic structure functions:

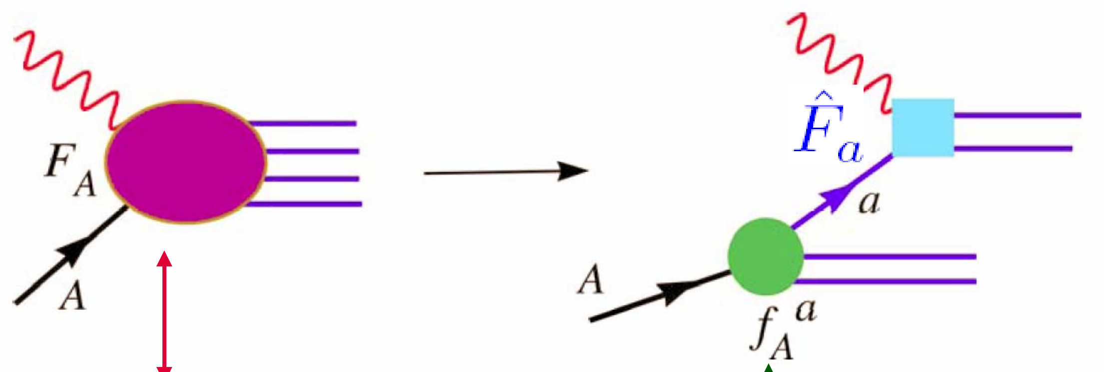
$$F_A^\lambda(x,Q) = \sum_a f_A^a(x,\mu) \otimes \hat{F}_a^\lambda(x,\frac{Q}{\mu}) \quad (a=q,g)$$

and that *the hadronic parton distributions* $f_A^a(x,\mu)$ *are finite*.

The three faces of the Magical Factorization Formula

#1 (seen in illustrative sample calculation)

Choose A to be a parton (say, A = b):



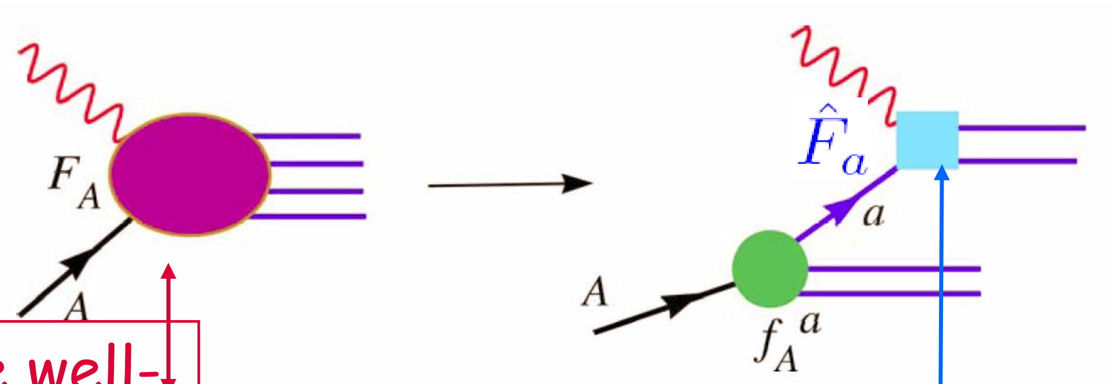
1. Calculate the partonic cross section F_b perturbatively (contains colinear and soft singularities).

2. Calculate the perturbative PDFs $f_b^{\ a}$ (also contains colinear and soft singularities)

Subtract (2) from (1), and derive the (IRS) hard cross sections $\hat{F}_a$ (Wilson coefficients).

The three faces of the Magical Factorization Formula #2

Choose F_A to be a set of established experimental cross sections



1. Input: the well-measured data for all $\{F_A\}$.

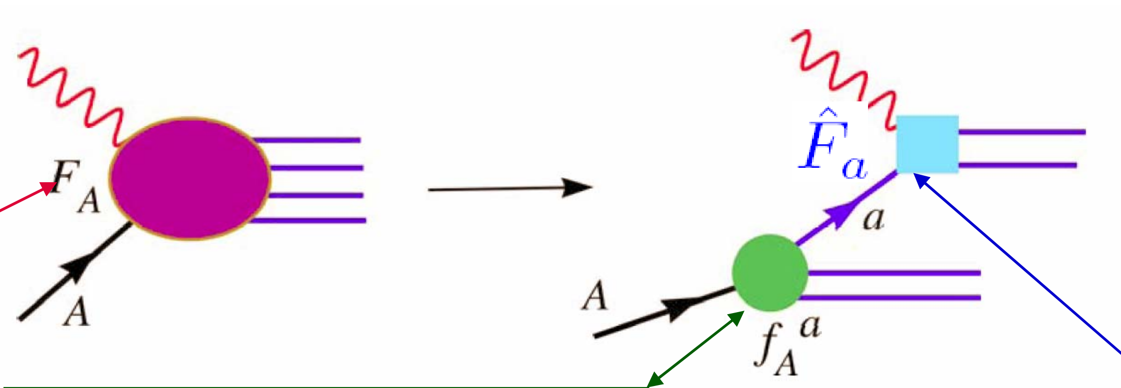
2. Input: known perturbatively calculated hard cross sections $\hat{F}_a$

Perform a Global QCD Analysis to determine the universal (but perturbatively non-calculable) hadronic parton distributions $f_A^a(x,\mu)$.

The three faces of the Magical Factorization Formula

#3

Choose F_A to be some physical cross section of interest:



1. Input hadronic PDFs determined by global QCD analysis .

2. Input: known perturbatively calculated hard cross sections $\hat{F}_a$

Predict cross sections for SM or New Physics Processes of interest!

The QCD Evolution Equation & Scale dependence of PDFs

Parton distributions represent long-distance physics,

- for parton targets, f_c^a are calculable, but contain collinear/soft "divergences";
- for hadron targets, f_A^a are finite, but not calculable in perturbation theory.

Since, f_c^a are calculable, one can examine their μ-dependence:

$$\frac{d}{d\ln\mu} f_c^a(x,\mu) = \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}(\frac{x}{\xi}, \alpha_s(\mu))\ f_c^b(\xi,\mu).$$

where P_{ab} on the RHS (the splitting functions) has the perturbative form

$$P_{ab}(z,\alpha_s(\mu)) = P_{ab}^{(1)}(z)\,\frac{\alpha_s(\mu)}{\pi} + P_{ab}^{(2)}(z)\,\left(\frac{\alpha_s(\mu)}{\pi}\right)^2 + \cdots .$$

Note, the splitting functions *$P_{ab}(x)$* are independent of the parent *c* , *because of factorization*.
Hence, the hadronic (i.e. physical) structure functions satisfy the *same QCD evolution equation*, with the *same evolution kernels* (splitting function) that are calculated using partonic amplitudes!

$$\frac{d}{d\ln\mu} f_A^a(x,\mu) = \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}\left(\frac{x}{\xi}, \alpha_s(\mu)\right) f_A^b(\xi,\mu)$$

Compare with the more familiar evolution equation for scale dependence of renormalization constants (e.g. $\alpha_s(\mu)$),

$$\mu \frac{d}{d\mu} Z_i(\mu) = \gamma_i Z_i(\mu)$$

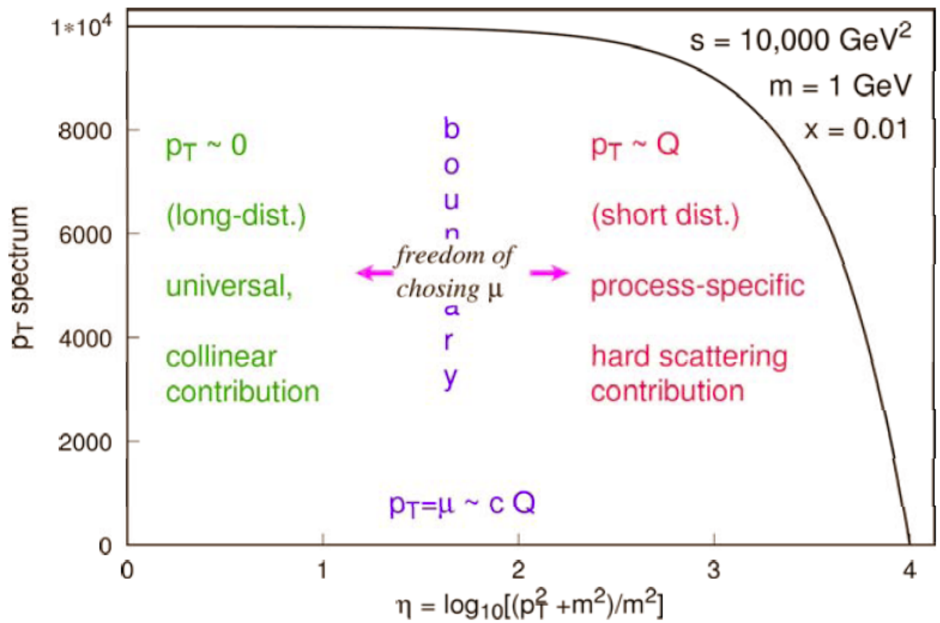
$P_{ab}(x)$ (calculable order by order in α_s) is the analogue of the *anormalous dimension* parameters γ_i that are characteristic of the underlying theory (QCD).

The Scale Dependence of Physical Predictions

$$F_A^\lambda(x,Q,m) = \sum_a f_A^a(x,\mu_F,m) \otimes \hat{F}_a^\lambda(x,\frac{Q}{\mu_F},\frac{m}{\mu_F},\alpha_x(\mu_R))$$

In principle, the choice of factorization scale μ_f should not affect physical predictions:

graphically,



analytically, in the perturbative approach:

$$\begin{aligned} \mu\frac{d}{d\mu}F &= \mu\frac{d}{d\mu}f \otimes \hat{F} + f \otimes \mu\frac{d}{d\mu}\hat{F} \\ &= O(\alpha_s^{N+1}) \end{aligned}$$

i.e. the physical F's are *scale independent*, to the order of the pert. calculation.

How to Choose the Renormalization/Factorization Scale?

- Similar considerations apply to μ_R and μ_F; to simplify the discussion, let $\mu_R = \mu_F = \mu$.

- In order to apply perturbative expansion, we need $\alpha_s(\mu)/\pi \ll 1 \Rightarrow \mu \gg \Lambda_{QCD}$.

- Since $\hat{F}_A$ contains terms of the form $\alpha_s^n(\mu)\ln^n(Q/\mu)$ (or even $\ln^{2n}(Q/\mu)$ in some cases), we cannot have $\mu \gg Q$ without spoiling the perturbative approach.

- Therefore, μ must be chosen to be of the order order as the hard scale Q, i.e. $\mu = cQ$ with $c \sim 1$.

- To estimate the uncertainty of a Nth order calculation, we often take ΔF_A to be the range of variation of F_A calculated with $\mu = cQ$ and $\frac{1}{2} < c < 1$—keeping in mind that $\mu\frac{d}{d\mu}F_A = O(\alpha_s^{N+1})$.

Scale Dependence of Physical Predictions - Example

Simplest Example: total cross section for $e^+e^- \to$ hadrons

$$\sigma_{\rm tot}(s) = \frac{12\pi\alpha^2}{s}\left(\sum_f Q_f^2\right)[1+\Delta] \qquad \text{(infra-red safe)}$$

$$\Delta = \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 C_2(\frac{\mu^2}{s}) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 C_3(\frac{\mu^2}{s}) + \cdots$$

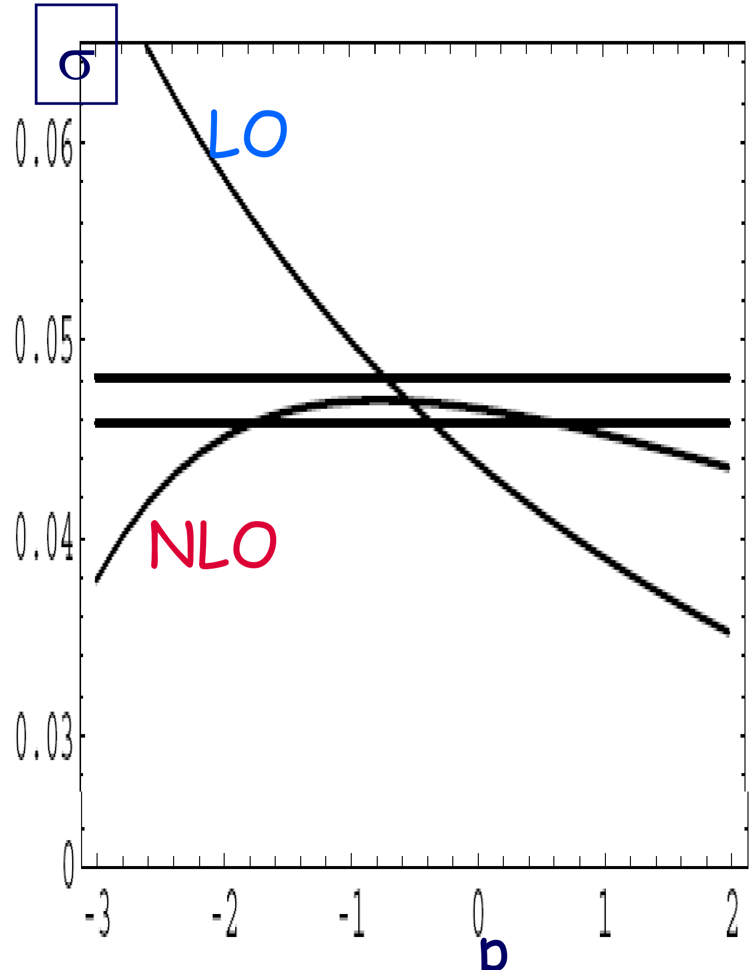
$$C_2(\frac{\mu^2}{s}) = 1.4092 + 1.9167\ \ln\left(\frac{\mu^2}{s}\right)$$

$$C_3(\frac{\mu^2}{s}) = -12.805 + 7.8186\ \ln\left(\frac{\mu^2}{s}\right) + 3.674\ \ln^2\left(\frac{\mu^2}{s}\right)$$

Scale dependence of e^+e^- cross section

LO term: scale dep. from the $\alpha_s^2(\mu)$ factor only—monotonic

NLO: compensation between the $\alpha_s^2(\mu)$ and the Δ correction factor leads to much smaller range of variation

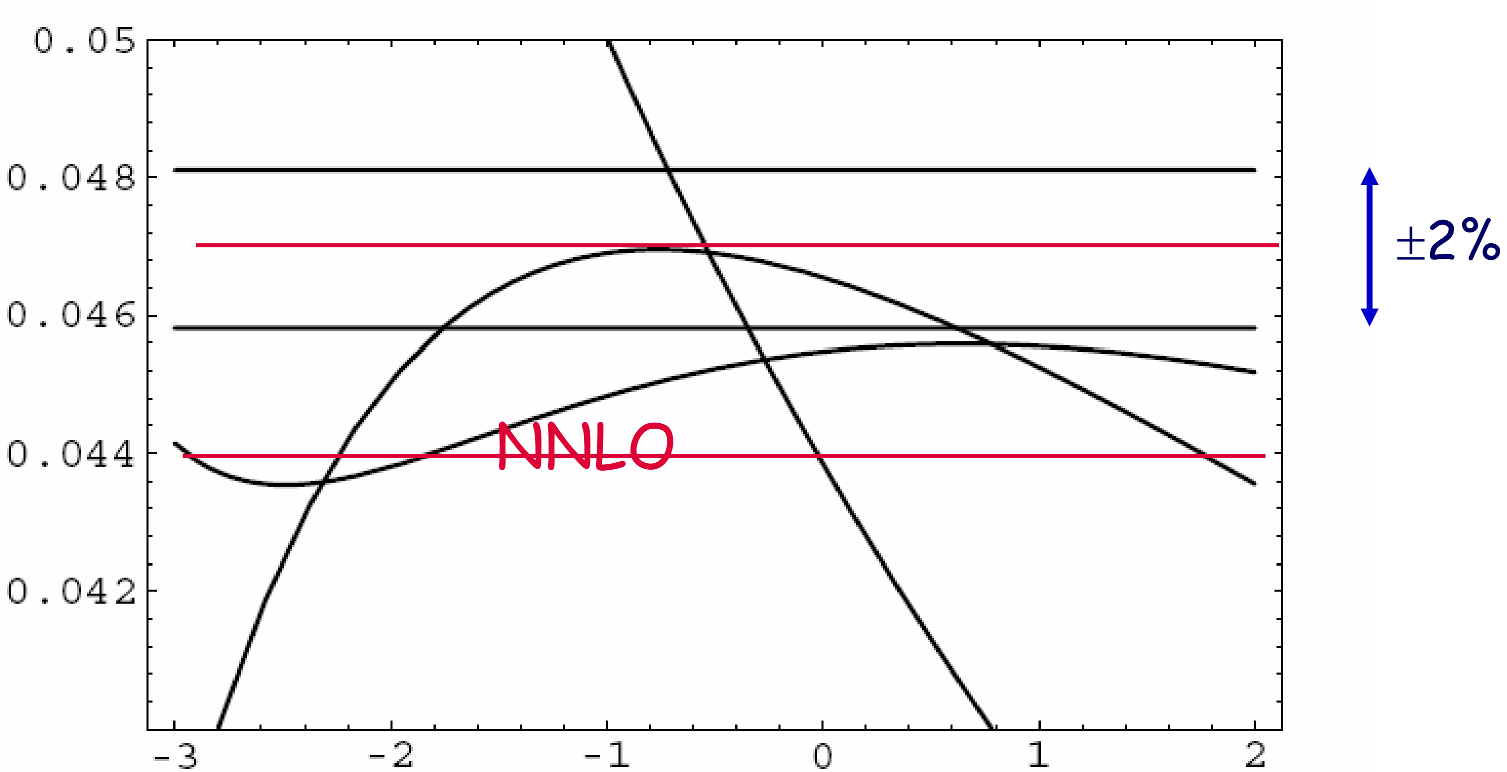


Error band:

Central value: $\hat{\mu}$ based on $\left[\frac{d\Delta_2(\mu)}{dln\mu}\right]|_{\mu=\hat{\mu}} = 0$

Error size: range of $\Delta(\mu)$ for $2\hat{\mu} > \mu > 0.5\hat{\mu}$,

Scale dependence of e^+e^- cross section



NNLO term reduces the scale dependence further, although somewhat outside the estimated error band—serve as a note of caution.

Diff between NLO and NNLO within the p(-2, 2) range is consistent with the estimated Δ in magnitude.