An Introduction to the Parton Model and Perturbative QCD

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Lecture IV - Hadron-hadron production of particles, jets, and photons

- Kinematics
- Observables in lowest order - QCD improved parton model
- Higher order corrections
- More complex observables and the need for Monte Carlo techniques
- Overview of phase space slicing methods

Pre-QCD

Berman, Bjorken, and Kogut, Phys. Rev. D4 (1971) 3388

- Gave predictions for the inclusive production of particles, jets (called "cores") and photons with large transverse momentum in hadron-hadron, lepton-hadron, and lepton-lepton processes
- Pre-dated QCD, but assumed an underlying DIS-type scattering process
- Predicted large yields at high-p_T
- Subsequently observed at the ISR and Fermilab

Post-QCD

- Asymptotic freedom suggested applicability of perturbative techniques
- Scale-dependent PDFs from DIS and lepton pair production
- Scale-dependent FFs from e^+e^- annihilation and also DIS particle production
- Use the "QCD-improved Parton Model" formalism described earlier
- High-p_T scattering would allow one to test the perturbative predictions for the hard scattering at the parton level
- Photon production would provide a "color blind" probe of the scattering
- The expectation that jets would be observed and that their 4-vectors could approximately be identified with those of the scattered partons was an important test of the picture

Inclusive jet production

- We will start with inclusive jet production process $A + B \to jet + X$ where A and B are hadrons and we will identify the jet with a scattered parton
- The basic cross section starts with an expression of the form

$$
\begin{aligned}
d\sigma(AB \to h + X) &= \frac{1}{2\hat{s}} \sum_{abcd} G_{a/A}(x_a, M_f^2)\, dx_a\, G_{b/B}(x_b, M_f^2)\, dx_b \\
&\quad \overline{\sum} |M_{ab\to cd}|^2\, dPS^{(2)}
\end{aligned}
$$

- We can simplify this expression if we evaluate the two-particle phase space factor using the Mandelstam variables

$$
\hat{s} = (p_a + p_b)^2 \qquad \hat{t} = (p_a - p_c)^2 \qquad \hat{u} = (p_b - p_c)^2
$$

- The two-body phase space factor is

$$dPS^{(2)} = \frac{d^3p_c}{(2\pi)^3 2E_c}\frac{d^3p_d}{(2\pi)^3 2E_d}(2\pi)^4\delta(p_a + p_b - p_c - p_d)$$

- Exercise: Show that $\frac{d^3p_d}{2E_d} = d^4p_d\delta(p_d^2)$ and that $\delta(p_d^2) = \delta(\hat{s} + \hat{t} + \hat{u})$ for massless partons and hence that

$$dPS^{(2)} = \frac{1}{8\pi^2}\frac{d^3p_c}{E_c}\delta(\hat{s} + \hat{t} + \hat{u})$$

- Parton 4-vectors in the hadron-hadron frame:

$$p_{\substack{a\\b}} = x_{\substack{a\\b}}(1, 0, 0, \pm 1) \quad p_{\substack{c\\d}} = p_T(\cosh y_{\substack{c\\d}}, 1, 0, \sinh y_{\substack{c\\d}})$$

- And in the parton-parton frame:

$$p_{\substack{a\\b}} = \frac{\sqrt{\hat{s}}}{2}(1, 0, 0, \pm 1) \quad p_{\substack{c\\d}} = \frac{\sqrt{\hat{s}}}{2}(1, \pm\sin\theta, 0, \pm\cos\theta)$$

- Exercise: Show that the Mandelstam variables in the hadron-hadron frame are

$$\hat{s} = x_a x_b s, \quad \hat{t} = -x_a \sqrt{s} p_T e^{-y_c}, \text{ and } \hat{u} = -x_b \sqrt{s} p_T e^{y_c}$$

and that in the the parton-parton frame they are

$$\hat{t} = -\frac{\hat{s}}{2}(1 - \cos\theta) \text{ and } \hat{u} = -\frac{\hat{s}}{2}(1 + \cos\theta)$$

- At the parton level one can write the cross section as

$$d\hat{\sigma} = \frac{1}{2\hat{s}} \overline{\sum} |M|^2 dPS^{(2)}$$

- It follows then that

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2} \overline{\sum} |M|^2$$

- Putting this all together one gets the following expression for the invariant cross section

$$
\begin{aligned}
E\frac{d^3\sigma}{d^3p}(A+B\to jet+X) &= \sum_{ab}\int G_{a/A}(x_a,M_f^2)\,dx_a G_{b/B}(x_b,M_f^2)\,dx_b \\
&\quad \frac{\hat{s}}{\pi}\frac{d\hat{\sigma}}{d\hat{t}}(ab\to cd)\delta(\hat{s}+\hat{t}+\hat{u})
\end{aligned}
$$

- Substitute in the expressions for the Mandelstam variables in the hadron-hadron frame and integrate on x_b to get

$$
\begin{aligned}
E\frac{d^3\sigma}{d^3p}(A+B\to jet+X) &= \sum_{ab}\int_{x_{amin}}^{1} dx_a\, G_{a/A}(x_a,M_f^2)G_{b/B}(x_b,M_f^2) \\
&\quad \frac{2}{\pi}\frac{x_a x_b}{2x_a - x_T e^{y_1}}\frac{d\hat{\sigma}}{d\hat{t}}(ab\to cd)
\end{aligned}
$$

where $x_b = \frac{x_a x_T e^{-y_1}}{2x_a - x_T e^{y_1}}$ and $x_{amin} = \frac{x_T e^{y_1}}{2 - x_T e^{-y_1}}$ corresponds to $x_b = 1$ and $x_T = \frac{2p_T}{\sqrt{s}}$.

Discussion

- Don't forget to sum over all possibilities for the parton-parton scattering
- For a gq initial state the gluon may come from hadron A and the quark form hadron B or the other way around! Both contributions are needed.
- Likewise, the jet may have come from parton c *or* from parton d. Again, both contributions must be included.
- When $y \approx 0$ then $x_a \sim x_b \sim x_T$
- Forward or backward rapidities are dominated by one x large and one x small

- There is still one integration which smears out a direct reconstruction of the underlying scattering
- Consider a dijet cross section where one measures the rapidities of the two scattered jets
- Easiest to start with the four-momentum conserving delta function in $dPS^{(2)}$
  - The transverse momentum parts ensure that the jets balance in p_T in lowest order
  - The energy and longitudinal delta functions determine both x_a and x_b

Dijet cross section

$$\frac{d\sigma}{dy_c dy_d dp_T^2} = \sum_{ab} x_a G_{a/A}(x_a, M_f^2)\, x_b G_{b/B}(x_b, M_f^2)\, \frac{d\hat{\sigma}}{d\hat{t}}(ab \to cd)$$

where

$$x_{\substack{a\\b}} = \frac{p_T}{\sqrt{s}}(e^{\pm y_c} + e^{\pm y_d})$$

- Using the expressions for the parton 4-vectors and the Mandelstam variables given earlier one can derive several useful relations

$$p_T^2 = \hat{t}\hat{u}/\hat{s} \qquad \cos\theta = (\hat{t} - \hat{u})/\hat{s}$$

- After a bit of work, these can be used to derive

$$\cos\theta = \tanh\frac{y_c - y_d}{2}$$

What about inclusive hadron production?

Insert a factor of $D_{h/c}(z, M_f^2)dz$ resulting in one more integration.

$$E\frac{d^3\sigma}{d^3p}(A+B\to h+X) = \sum_{abc}\int_{x_{amin}}^{1} dx_a \int_{x_{bmin}}^{1} dx_b\, G_{a/A}(x_a, M_f^2) G_{b/B}(x_b, M_f^2) D_{h/c}(z, M_f^2)\frac{1}{\pi z}\frac{d\hat{\sigma}}{d\hat{t}}(ab\to cd)$$

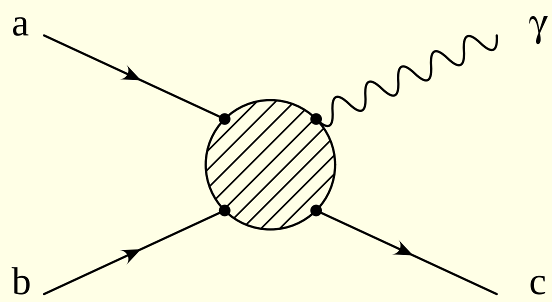
where $z = \frac{x_T}{2x_b}e^{-y} + \frac{x_T}{2x_a}e^{y}$.

Also, $x_{bmin} = \frac{x_a x_T e^{-y}}{2x_a - x_T e^{y}}$ and $x_{amin} = \frac{x_T e^{y}}{2 - x_T e^{-y}}$.

- Note that the hadron 4-vector is given by $p = zp_c$
- One must also include the contribution where parton d fragments into the observed hadron

Direct Photon Production - Theory Overview

- Lowest Order: $\mathcal{O}(\alpha\alpha_s)$
  1. $qg \to \gamma q$ QCD Compton
  2. $q\bar{q} \to \gamma g$ annihilation
- The single photon invariant cross section is given by a convolution with the beam and target parton distribution functions



$$d\sigma(AB \to \gamma + X) = G_{a/A}(x_a, \mu_F)\, dx_a\, G_{b/B}(x_b, \mu_F)\, dx_b \frac{1}{2\hat{s}} \overline{\sum_{ab}} |M(ab \to \gamma c)|^2 dPS^{(2)}$$

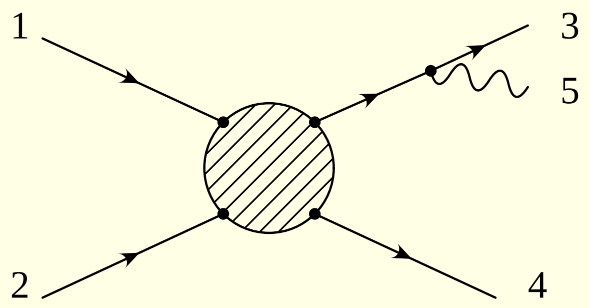
- Event topology is that of an isolated photon recoiling against a jet (either a quark or a gluon)

Next-to-Leading Order: $\mathcal{O}(\alpha\alpha_s^2)$

1. one-loop virtual contributions
2. $q\overline{q} \to \gamma gg$
3. $gq \to \gamma qg$
4. $qq' \to \gamma qq'$ plus related subprocesses

- In the next order one sees a new configuration wherein the photon is no longer isolated. Instead, it may be radiated off a high-p_T quark produced in the hard scattering process.

- Consider the subprocess $q(1)q(2) \to q(3)q(4)\gamma(5)$
- Examine the region where $s_{35} = (p_3 - p_5)^2 \approx 0$



$$\overline{\sum}|M(qq \to qq\gamma)|^2 \approx \frac{\alpha}{2\pi} \, P_{\gamma q}(z) \, \frac{1}{s_{35}} \, \overline{\sum}|M(qq \to qq)|^2$$

- An internal quark line is going on-shell signalling long distance physics effects
- Gives rise to a collinear singularity
- Can factorize the singularity by introducing a *photon fragmentation function*

Photon Fragmentation

- Photon is accompanied by jet fragments on the *same* side
- Factorize the singularity and include it in the bare photon fragmentation function
- Sum large logs with modified Altarelli-Parisi equations

$$
\begin{aligned}
Q^2 \frac{dD_{\gamma/q}(x,Q^2)}{dQ^2} &= \frac{\alpha}{2\pi} P_{\gamma q} + \frac{\alpha_s}{2\pi} \left[D_{\gamma/q} \otimes P_{qq} + D_{\gamma/g} \otimes P_{gq} \right] \\
Q^2 \frac{dD_{\gamma/g}(x,Q^2)}{dQ^2} &= \frac{\alpha_s}{2\pi} \left[\sum_q D_{\gamma/q} \otimes P_{qg} + D_{\gamma/g} \otimes P_{gg} \right]
\end{aligned}
$$

- As with hadron PDFs and fragmentation functions, can't perturbatively calculate the fragmentation functions, but the scale dependence is perturbatively calculable
- Note the $P_{\gamma q}$ splitting function - represents the pointlike coupling of the photon to the quark in $q \to \gamma q$

Fragmentation Component

- The situation has become more complex
- Expect to see two classes of events

  1. Direct (or pointlike) - no hadrons accompanying the photon
  2. Fragmentation (or bremsstrahlung) - photon is a fragment of a high-p_T jet. Part of the fragmentation function is perturbatively calculable.

- Expect (1) to dominate at high-p_T since the energy is not shared with accompanying hadrons.
- The $P_{\gamma q}$ splitting function gives rise to the leading high Q^2 behavior going as $\alpha \log(Q^2/\Lambda^2) \sim \frac{\alpha}{\alpha_s}$ (see Appendix II for a derivation)

So, to our list of contributions add those involving photon fragmentation functions

- $\mathcal{O}(\alpha\alpha_s) : \frac{d\sigma}{d\hat{t}}(ab \to cd) \otimes D_{\gamma/c}$
- $\mathcal{O}(\alpha\alpha_s^2) : \frac{d\sigma}{d\hat{t}}(ab \to cde) \otimes D_{\gamma/c}$

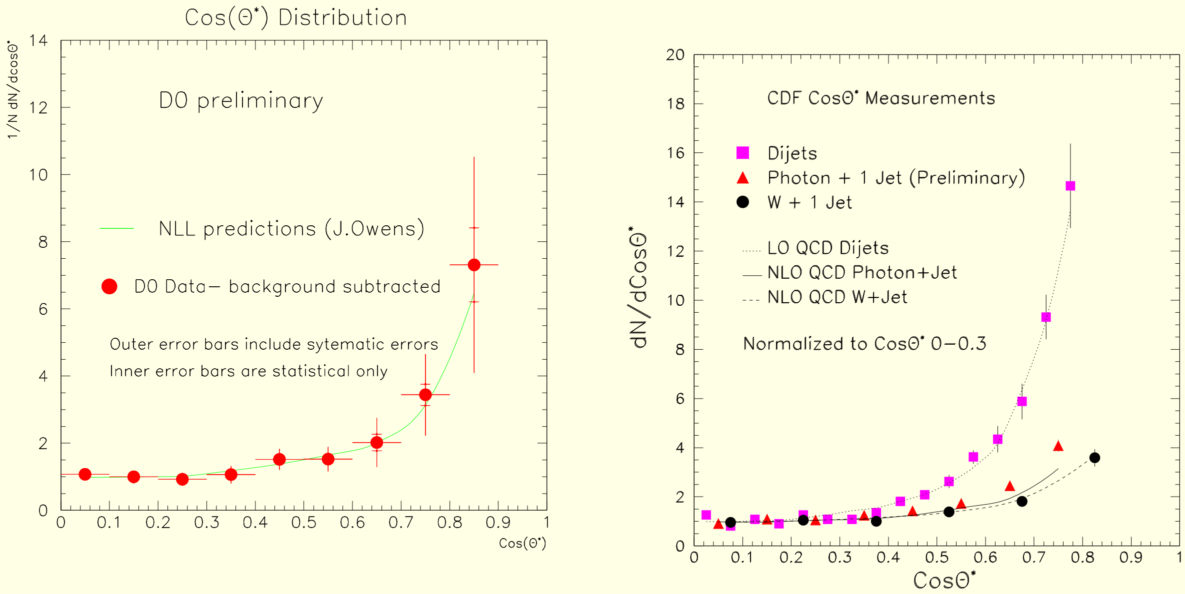
Some Comments

- Photons can be produced as fragments of jets, as is also the case for particles
- Photon production therefore involves all of the subprocesses relevant for jet or particle production
- In addition, one also has the pointlike production processes

Photon production is *more* complicated than jet production, not *less*

Angular Dependence

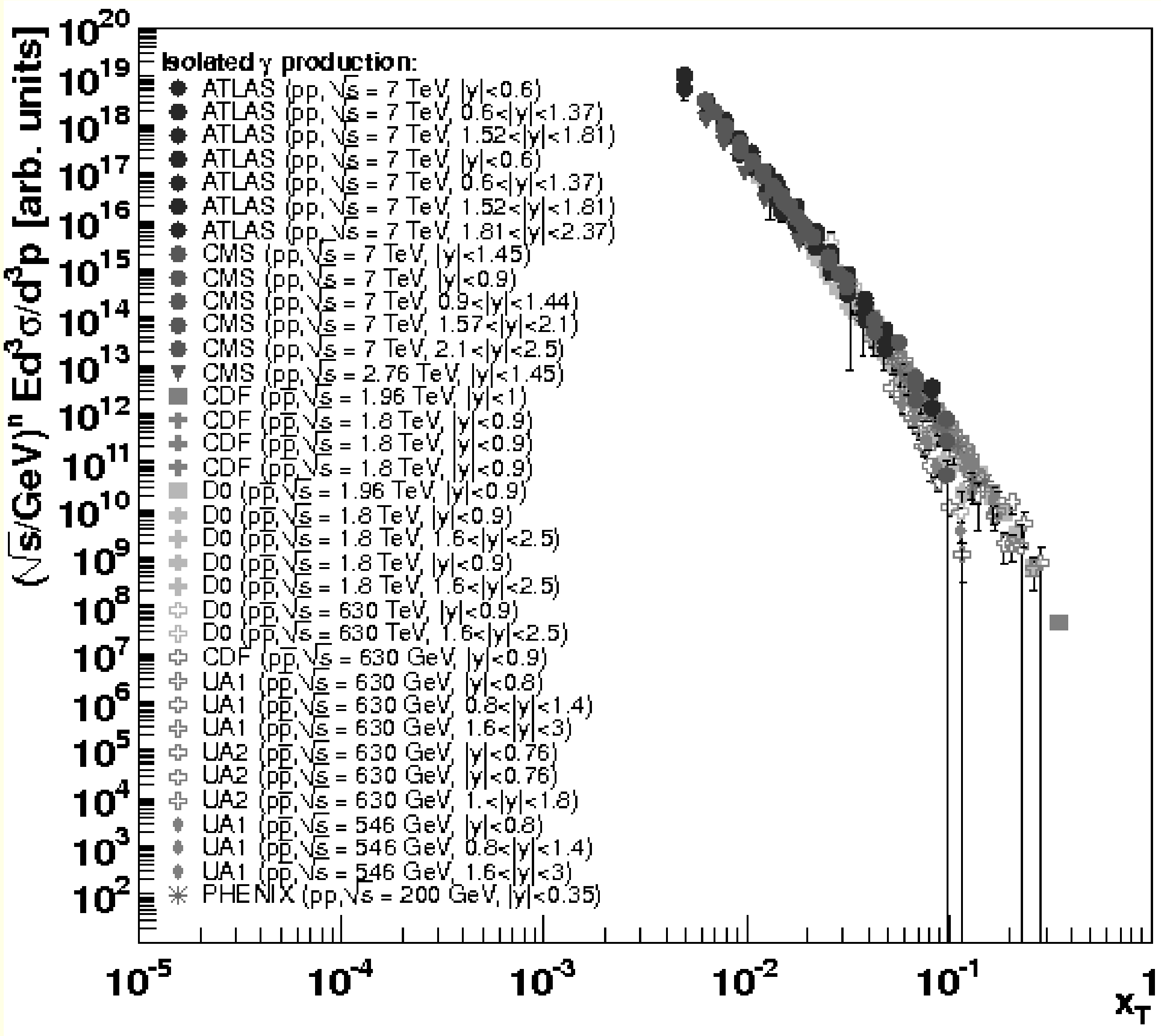
- Appendix I lists the expressions for the two-body QCD subprocesses
- t channel vector boson exchange: $\frac{1}{\hat{t}^2} \sim \frac{1}{(1-\cos\theta)^2}$ (dominant in jet production)
- t channel fermion exchange: $\frac{1}{\hat{t}} \sim \frac{1}{(1-\cos\theta)}$ (dominant in direct photon production)
- Can measure the angular distributions and see the differences between jets and γs



The different angular distributions for jet, photon, and weak boson production are well described by QCD

Scaling Behavior

- The invariant cross section $E\frac{d^3\sigma}{dp^3}$ has mass dimension 4. Thus, at fixed values of x_T and θ (or rapidity) it should scale as p_T^{-4}
- Plot data versus x_T
- Data should lie on a common curve if multiplied by p_T^4
- Compilation of isolated direct photon data from d'Enterria and Rojo (arXiv:1202.1762[hep-ph])
- Idea works, but n=4.5, not 4 – Why?



Isolated γ production:
ATLAS (pp, √s = 7 TeV, |y|<0.6)
ATLAS (pp, √s = 7 TeV, 0.6<|y|<1.37)
ATLAS (pp, √s = 7 TeV, 1.52<|y|<1.81)
ATLAS (pp, √s = 7 TeV, |y|<0.6)
ATLAS (pp, √s = 7 TeV, 0.6<|y|<1.37)
ATLAS (pp, √s = 7 TeV, 1.52<|y|<1.81)
ATLAS (pp, √s = 7 TeV, 1.81<|y|<2.37)
CMS (pp, √s = 7 TeV, |y|<1.45)
CMS (pp, √s = 7 TeV, |y|<0.9)
CMS (pp, √s = 7 TeV, 0.9<|y|<1.44)
CMS (pp, √s = 7 TeV, 1.57<|y|<2.1)
CMS (pp, √s = 7 TeV, 2.1<|y|<2.5)
CMS (pp, √s = 2.76 TeV, |y|<1.45)
CDF (pp̄, √s = 1.96 TeV, |y|<1)
CDF (pp̄, √s = 1.8 TeV, |y|<0.9)
CDF (pp̄, √s = 1.8 TeV, |y|<0.9)
CDF (pp̄, √s = 1.8 TeV, |y|<0.9)
D0 (pp̄, √s = 1.96 TeV, |y|<0.9)
D0 (pp̄, √s = 1.8 TeV, |y|<0.9)
D0 (pp̄, √s = 1.8 TeV, 1.6<|y|<2.5)
D0 (pp̄, √s = 1.8 TeV, |y|<0.9)
D0 (pp̄, √s = 1.8 TeV, 1.6<|y|<2.5)
D0 (pp̄, √s = 630 TeV, |y|<0.9)
D0 (pp̄, √s = 630 TeV, 1.6<|y|<2.5)
CDF (pp̄, √s = 630 GeV, |y|<0.9)
UA1 (pp̄, √s = 630 GeV, |y|<0.8)
UA1 (pp̄, √s = 630 GeV, 0.8<|y|<1.4)
UA1 (pp̄, √s = 630 GeV, 1.6<|y|<3)
UA2 (pp̄, √s = 630 GeV, |y|<0.76)
UA2 (pp̄, √s = 630 GeV, |y|<0.76)
UA2 (pp̄, √s = 630 GeV, 1.<|y|<1.8)
UA1 (pp̄, √s = 546 GeV, |y|<0.8)
UA1 (pp̄, √s = 546 GeV, 0.8<|y|<1.4)
UA1 (pp̄, √s = 546 GeV, 1.6<|y|<3)
PHENIX (pp, √s = 200 GeV, |y|<0.35)
(√s/GeV)^n Ed³σ/d³p [arb. units]
x_T

The explanation is due to the various sources of scaling violations in the theory

- α_s depends on the dimensionless quantity $\ln(\mu/\Lambda)$
- Typically one expects $\mu \approx p_T$
- As p_T increases α_s decreases and one needs to multiply the cross section by a higher power of p_T in order to get the scaling curve
- Similarly, the PDFs above $x \approx .2$ decrease with increasing p_T causing an additional increase in the power of p_T required to get the data to lie on a common curve
- Similar behavior exists for jets since the factors of the PDFs and α_s are the same
- A slightly higher power is seen for inclusive hadron production since one has an additional scale-violating fragmentation function

Now, let's look at the form of the NLO corrections

By now, we can anticipate what needs to be done

- Use the two-loop running coupling
- Use NLO PDFs
- Use $2 \to 3$ tree-level matrix elements and $2 \to 2$ loop corrections
- Use three particle phase space for the tree graphs
- Use dimensional regularization
- Factorize the collinear singularities associated with the PDFs and FFs
- This will result in an $\mathcal{O}(\alpha_s)$ correction to the preceding expression for the invariant cross section

- Generally, the high-p_T calculation outlined above is appropriate for problems where there is one large scale
- This means that $\hat{s} \sim \hat{t} \sim \hat{u}$
- In terms of the kinematic parameters of the observed hadron, this means that the rapidity is not near the edge of phase space and that the transverse momentum is large, *i.e.*, $x_T = \frac{2p_T}{\sqrt{S}}$ is neither near one nor near zero
- In this region the NLO calculation will generally provide
  - Reduced scale dependence
  - A modest correction to the normalization of the lowest order result
- There may be some changes in the p_T and y distributions due to the presence of new subprocesses
- Note that in this example we have integrated over both of the recoiling partons. This smooths out regions where there might otherwise have been large corrections.
- Next, let's look at how the reduction in scale dependence actually comes about and how we might use this to our advantage

Scale dependence

- Consider a highly simplified example of jet production in hadron-hadron scattering at large enough values of x_T that only valence quark scattering subprocesses need be considered.
- Denote the lowest order result for the invariant cross section by

$$E\frac{d^3\sigma}{dp^3} \equiv \sigma = \alpha_s^2(\mu)\hat{\sigma}_B \otimes q(M) \otimes q(M)$$

- Here $\hat{\sigma}_B$ denotes the lowest order parton-parton scattering cross section while $q(M)$ denotes a quark PDF with factorization scale M
- I have separated out the running coupling which is evaluated at a renormalization scale μ
- $\otimes$ denotes a convolution

$$f \otimes g = \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y)$$

- With this same notation the NLO calculation will have the form

$$\begin{aligned}
\sigma &= \alpha_s^2(\mu)\hat{\sigma}_B \otimes q(M) \otimes q(M) \\
&+ 2b\alpha_s^3(\mu)\ln\frac{\mu^2}{p_T^2}\hat{\sigma}_B \otimes q(M) \otimes q(M) \\
&+ 2\frac{\alpha_s^3(\mu)}{2\pi}\ln\frac{p_T^2}{M^2}P_{qq} \otimes q(M) \otimes q(M) \\
&+ \alpha_s^3(\mu)K \otimes q(M) \otimes q(M)
\end{aligned}$$

- I have separated out the parts of the NLO correction which contain explicit logs of μ or of M and have normalized then using p_T
- K denotes the remainder of the NLO correction

Recall that

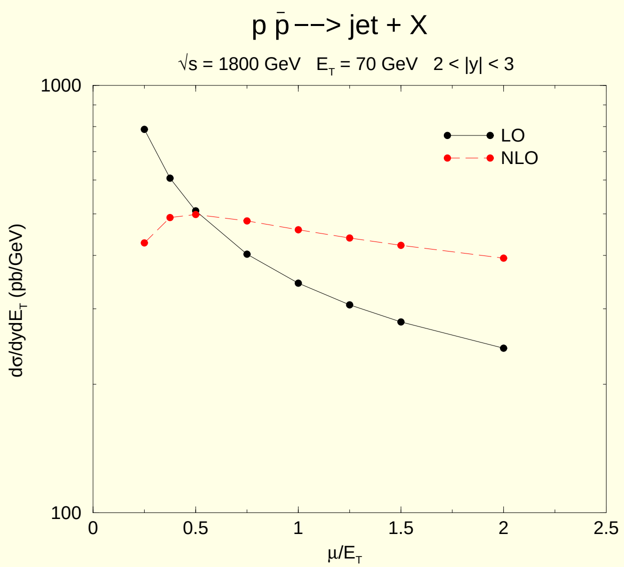
$$\mu^2 \frac{\partial \alpha_s(\mu)}{\partial \mu^2} = -b\alpha_s^2 + \ldots$$

and that the nonsinglet PDF satisfies

$$M^2 \frac{\partial q(x,M)}{\partial M^2} = \frac{\alpha_s}{2\pi} P_{qq} \otimes q(M)$$

- Now, calculate $\mu^2 \frac{\partial \sigma}{\partial \mu^2}$
- The derivative of the first line gives a contribution which cancels a piece of the derivative of the second line; the remaining derivatives of the second, third, and fourth lines all give contributions of $\mathcal{O}(\alpha_s^4)$
- The μ dependence is thus zero to $\mathcal{O}(\alpha_s^3)$ (Exercise: Fill in the steps to show this)

- Now, calculate $M^2 \frac{\partial \sigma}{\partial M^2}$
- Again, the derivative of the first line cancels a portion of the derivative of the third and the remaining derivatives give results of $\mathcal{O}(\alpha_s^4)$ (Exercise: Fill in the steps to show this)
- Both the renormalization and factorization scale dependences cancel to the order calculated, although there is still residual scale dependence due to higher order corrections
- The following plot shows the type of behavior which is typical of LO and NLO calculations



Understanding the scale dependences

- To simplify the discussion, consider the situation where $\mu = M$, as in the previous plot
- For the lowest order calculation we understand that increasing μ causes the running coupling to decrease
- In the region of $x \gtrsim .1$ an increase of M also causes the PDFs to decrease - this is the region relevant for our high-p_T jet example
- Thus, the LO calculation is a monotonically decreasing function of the scale
- For the full NLO calculation, the first line (lowest order result) and the last line (residual NLO result) both have the same type of monotonically decreasing behavior as the scale increases

$$\begin{aligned}\sigma &= \alpha_s^2(M)\hat{\sigma}_B \otimes q(M) \otimes q(M) + \ldots \\ &+ \alpha_s^3(M) K \otimes q(M) \otimes q(M)\end{aligned}$$

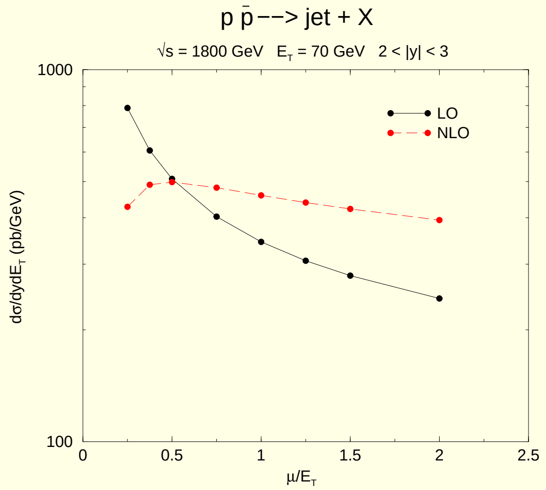
- The $\ln \frac{M^2}{p_T^2}$ factor in the second line causes this contribution to be negative for $M < p_T$ and to be positive once M exceeds p_T

$$\begin{aligned} \sigma &= \ldots \\ &+ 2b\alpha_s^3(M) \ln \frac{M^2}{p_T^2} \hat{\sigma}_B \otimes q(M) \otimes q(M) + \ldots \end{aligned}$$

- For the third line, recall that the convolution with the splitting function gives a negative contribution in the region of interest since the slope of the scaling violations is negative there

$$\begin{aligned} \sigma &= + \ldots \\ &+ 2\frac{\alpha_s^3(M)}{2\pi} \ln \frac{p_T^2}{M^2} P_{qq} \otimes q(M) \otimes q(M) + \ldots \end{aligned}$$

- Thus, for $M < p_T$ the third line is negative and it turns positive for $M > p_T$
- The explicit logs in lines 2 and 3 thus cause the NLO curve to be below the LO curve if $\mu = M < p_T$ and to be above it if the scales are greater than p_T, as shown in the plot



- Note that this is an approximate argument and that there can be exceptions to it caused, for example by new channels opening in higher order
- The exact crossover point depends on the relative sizes of the contributions from each of the four lines

"Which scale is best?"

There are various methods for guestimating the choice of scale in NLO processes. Here are some examples:

- Principle of Minimal Sensitivity
  - An exact calculation would have no scale dependence - perturbative calculations are incomplete
  - The PMS scheme enforces $\frac{\partial\sigma}{\partial\mu} = \frac{\partial\sigma}{\partial M} = 0$
  - In my example the full dependence on μ was displayed so one can solve for the value of μ which makes the derivative zero - not the same as having it be zero to $\mathcal{O}(\alpha_s^3)$
  - Similarly, can numerically solve for the value of M which forces $\frac{\partial\sigma}{\partial M}$ to be zero
  - For each kinematic point the plot of μ versus M gives a saddle point structure and one can read off the correct values for both scales

- Could also use a "1-scale" PMS scheme - then one can read off the optimal scale from plots like the one shown previously which suggested $\mu = M \approx p_T/2$
- Method of Fastest Apparent Convergence
  - Choose the scale such that NLO and LO calculations are equal
  - All the higher order corrections are effectively absorbed into the logs of the scales

There is no unique prescription for choosing the scales

- When the higher order corrections are under control, both schemes give similar results of the order of the single large scale in the process
- In the plot shown earlier both schemes would suggest the choice of $p_T/2$ which is close to the "natural" choice of p_T

The ratio of the NLO and LO curves is just the K factor. It is obviously very scale dependent. For the FAC scheme the K factor is *defined* to be 1!

Comments

Does the scale dependence always work the way I have described? No!

- A new channel can open up in the next order that has no counterpart at the Born level - it is a new contribution, not just a correction to the LO result
- Example: Heavy quark production

  LO: $q\bar{q} \to Q\bar{Q} \quad gg \to Q\bar{Q}$
- Here the heavy quarks appear on opposite sides of the events - the heavy quark jets balance in p_T

  NLO $gg \to Q\bar{Q}g + \ldots$

  Here is a new configuration where a pair of heavy quarks recoils against a gluon jet
- This provides a large contribution whose scale dependence is not compensated by an LO contribution

Lesson: look at the subprocesses involved. If a new channel opens up, the next order corrections *may not* reduce the scale dependence.

More complicated observables

For the single particle cross section it is easy to calculate both the p_T and y distributions. However, there are other interesting observables

- Jets - one needs to be able to form jets of according to some jet definition which may not be easy to express in terms of partonic variables
- One might wish to examine joint distributions involving more than one particle
- Classic example - one might wish to calculate or measure the angular distribution of the scattered partons in their center of mass frame. With $2 \to 3$ subprocesses, how do you define this?
- One might wish to place cuts (kinematic constraints) on the final state particles. Sometimes this is easy (cuts in p_T or y)
- Sometimes the Jacobian between the experimentally observed variables and the parton level variables can not be easily calculated
- Suggests using a Monte Carlo formalism so that the cuts can be made on an event-by-event basis
- But what about the divergent terms?

Next-to-Leading-Order Calculations – Recap

- Ingredients
  - $2 \rightarrow 2$ $\mathcal{O}(\alpha_s^2)$ subprocesses, *e.g.*, $qq \rightarrow qq, qg \rightarrow qg$, and $gg \rightarrow gg$
  - $\mathcal{O}(\alpha_s^3)$ one-loop corrections to $2 \rightarrow 2$ subprocesses
  - $2 \rightarrow 3$ subprocesses such as $qq \rightarrow qqg$, etc
- $\mathcal{O}(\alpha_s^3)$ terms have singular regions corresponding to soft gluons and/or collinear partons
- Need a method to handle such singularities
- Observables involve many kinematic variables since we are interested in going beyond the single particle case
- Jacobians from parton variables to hadron variables are complex
- Suggests using a Monte Carlo approach, but one which allows the singularities to be properly treated

Two basic techniques

1. Phase space slicing
2. Dipole subtraction

- The ideas behind both are similar - the singular regions are at the edge of the 3-body phase space where the configurations look that those of 2-body phase space
- Integrate over the singular regions and combine the result with the 2-body singular terms
- factorize the collinear singularities associated with PDFs and FFs as before
- The result will be a set of finite expressions for the 2-body subprocesses and a second set for the 3-body processes
- These can be added together in appropriate histograms for various observables
- Modern applications allow such NLO calculations to be embedded in Monte Carlo shower programs

- Such programs will be covered in later lectures in this school
- See Appendix III for more on phase space slicing

Conclusions

- In these four lectures I've shown how the parton model emerged to provide a description of lepton-lepton, lepton-hadron, and hadron-hadron processes
- QCD in the leading-logarithm approximation modifies the parton model with the introduction of the running coupling α_s and scale violating PDFs and FFs that satisfy the appropriate DGLAP Equations
- Furthermore, QCD specifies how to calculate NLO corrections and beyond
- Later lectures in this Summer School will build on the foundation described in these lectures and show how QCD provides an excellent description of high energy large momentum transfer processes

Appendix I: Two-body QCD Subprocesses

Subprocess	$\frac{d\hat{\sigma}}{d\hat{t}}$ (in units of $\pi\alpha_s^2/\hat{s}^2$)
$qq' \to qq'$	$\frac{4}{9}\,\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$
$qq \to qq$	$\frac{4}{9}\left[\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}+\frac{\hat{s}^2+\hat{t}^2}{\hat{u}^2}\right]-\frac{8}{27}\,\frac{\hat{s}^2}{\hat{t}\hat{u}}$
$q\bar{q} \to q'\bar{q}'$	$\frac{4}{9}\,\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}$
$q\bar{q} \to q\bar{q}$	$\frac{4}{9}\left[\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}+\frac{\hat{u}^2+\hat{t}^2}{\hat{s}^2}\right]-\frac{8}{27}\,\frac{\hat{u}^2}{\hat{s}\hat{t}}$
$gq \to gq$	$-\frac{4}{9}\left[\frac{\hat{s}}{\hat{u}}+\frac{\hat{u}}{\hat{s}}\right]+\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$
$q\bar{q} \to gg$	$\frac{32}{27}\left[\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}\right]-\frac{8}{3}\,\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}$
$gg \to q\bar{q}$	$\frac{1}{6}\left[\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}\right]-\frac{3}{8}\,\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}$
$gg \to gg$	$\frac{9}{2}\left[3-\frac{\hat{t}\hat{u}}{\hat{s}^2}-\frac{\hat{s}\hat{u}}{\hat{t}^2}-\frac{\hat{s}\hat{t}}{\hat{u}^2}\right]$

Subprocess	$\frac{d\hat{\sigma}}{d\hat{t}}$ (in units of $\pi\alpha\alpha_s/\hat{s}^2$)
$gq \to \gamma q$	$-\frac{1}{3}e_q^2\left[\frac{\hat{u}}{\hat{s}}+\frac{\hat{s}}{\hat{u}}\right]$
$q\bar{q} \to \gamma g$	$\frac{8}{9}e_q^2\left[\frac{\hat{u}}{\hat{t}}+\frac{\hat{t}}{\hat{u}}\right]$

Appendix II - Photon fragmentation functions

- Rewrite the evolution equations by taking moments of both sides using the following definitions:

$$
\begin{aligned}
M_q^n &= \int_0^1 dx\ x^{n-1}\ D_{\gamma/q}(x) \\
M_g^n &= \int_0^1 dx\ x^{n-1}\ D_{\gamma/g}(x) \\
A_{ij}^n &= \frac{1}{2\pi b}\int_0^1 dx\ x^{n-1}\ P_{ij}(x) \\
a^n &= \frac{\alpha}{2\pi}\int_0^1 dx\ x^{n-1}\ P_{\gamma q} \\
\alpha_s(t) &= \frac{1}{bt}
\end{aligned}
$$

where $t = \ln(Q^2/\Lambda^2)$.

- The evolution equations can now be written as

$$\frac{dM_q^n}{dt} = e_q^2 \, a^n + \frac{1}{t} \left[A_{qq}^n \; M_q^n + A_{gq}^n \; M_g^n\right]$$

$$\frac{dM_g^n}{dt} = \frac{1}{t} \left[\sum_q A_{qg}^n M_q^n + A_{gg}^n M_g^n\right]$$

- If each of the moments is proportional to t, the t dependence drops out of the equations and they may be solved algebraically

- The asymptotic solution is

$$
\begin{aligned}
M_q^n &= a^n \left(\frac{e_q^2 - 5/18}{1 - A_{qq}^n} + \frac{5}{18} \frac{1 - A_{gg}^n}{F^n} \right) t \\
M_g^n &= \frac{5f}{9} a^n \frac{A_{gq}^n}{F^n} t \\
F^n &= 1 - A_{qq}^N - A_{gg}^n + A_{qq}^n A_{gg}^n - 2f A_{qg}^n A_{gq}^n
\end{aligned}
$$

where f is the number of flavors

- Note how the moments are each proportional to t
- Compare to the case where $P_{q\gamma} = 0$ where the moments are of the form

$$
M^n(t_0) \left(\frac{t}{t_0} \right)^{A^n}
$$

- Note that one can add any solution of the homogeneous evolution equations to this asymptotic solution

Appendix III – Phase Space Slicing Monte Carlo

- See B. Harris and J.F. Owens hep-ph/0102128
- Work in n=4-2ϵ dimensions using dimensional regularization
- Notation:
  - At the parton level: $p_1 + p_2 \to p_3 + p_4 + p_5$
  - Let $s_{ij} = (p_i + p_j)^2$ and $t_{ij} = (p_i - p_j)^2$
- Partition $2 \to 3$ phase space into three regions
  1. Soft: gluon energy $E_g < \delta_s \sqrt{s_{12}}/2$
  2. Collinear: s_{ij} or $|t_{ij}| < \delta_c s_{12}$
  3. Finite: everything else

- In soft region use the soft gluon approximation to generate a simple expression for the squared matrix element which can be integrated by hand
- In the collinear region use the leading pole approximation to generate a simple expression which can be integrated by hand.
- Resulting expressions have explicit poles from soft and collinear singularities
- Factorize initial and final state mass singularities and absorb into the fragmentation and distribution functions
- Add soft and collinear integrated results to the $2 \rightarrow 2$ contributions – singularities cancel
- Generate finite region contributions in 4 dimensions using usual Monte Carlo techniques
- End results is a set of two-body weights and a set of three-body weights.
- Both are finite and both depend on the cutoffs δ_s and δ_c
- Cutoff dependence cancels for sufficiently small cutoffs when the two sets of weights are added at the histogramming stage

Simple Example

Consider the integral of a quantity which has a pole at $x = 0$. Using dimensional regularization, one has an integral of the form

$$F = \int_0^1 dx\, x^{-1-\epsilon} f(x).$$

For x very near zero, approximate $f(x)$ by $f(0)$ yielding

$$F \approx f(0) \int_0^\delta dx\, x^{-1-\epsilon} + \int_\delta^1 dx\, x^{-1-\epsilon} f(x)$$

The first integral can be done analytically. The second is finite and can be evaluated with $\epsilon = 0$.

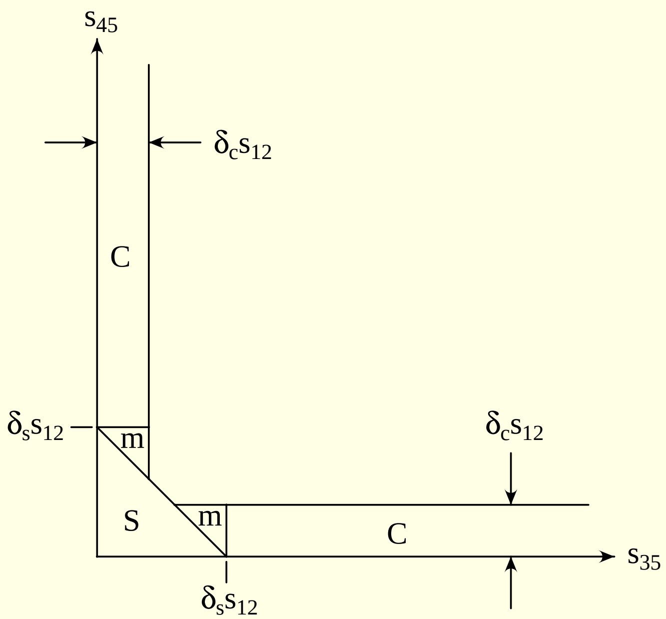
$$F \approx -\frac{f(0)}{\epsilon} + f(0) \log \delta + \int_\delta^1 dx\, \frac{f(x)}{x}.$$

The second integral can be done numerically. The dependence on the cutoff δ cancels for sufficiently small values of δ

Another Simple Example

Consider the example from Lecture I of the total cross section for $e^+e^- \rightarrow hadrons$. The complete first order QCD correction is simply $\frac{\alpha_s}{\pi}$.

- Phase space can be written in terms of two variables. It is convenient to choose these to be s_{35} and s_{45}.



- This sketch shows the soft, hard collinear, and finite regions of phase space

- The 1-loop virtual corrections lie at the origin in the lower left and are included in the soft region
- In the soft region the squared matrix element takes on a relatively simple form which may be integrated to yield

$$d\sigma_{\mathrm{S}} = d\sigma^0 \left[\frac{\alpha_s}{2\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu_r^2}{s_{12}} \right)^{\epsilon} \right] \left(\frac{A_2^s}{\epsilon^2} + \frac{A_1^s}{\epsilon} + A_0^s \right)$$

with

$$\begin{aligned} A_2^s &= 2C_F \\ A_1^s &= -4C_F \ln \delta_s \\ A_0^s &= 4C_F \ln^2 \delta_s \end{aligned}$$

- The final state hard collinear contribution can be simplified in the collinear region and easily integrated to yield

$$d\sigma_{\mathrm{HC}}^{q\to qg} = d\sigma^0 \left[\frac{\alpha_s}{2\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu_r^2}{s_{12}} \right)^{\epsilon} \right] \left(\frac{A_1^{q\to qg}}{\epsilon} + A_0^{q\to qg} \right)$$

with

$$A_1^{q\to qg} = C_F\,(3/2 + 2\ln\delta_s)$$
$$A_0^{q\to qg} = C_F\left[7/2 - \pi^2/3 - \ln^2\delta_s - \ln\delta_c\,(3/2 + 2\ln\delta_s)\right]$$

- The virtual contribution is given by

$$d\sigma_{\rm V} = d\sigma^0\left[\frac{\alpha_s}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu_r^2}{s_{12}}\right)^{\epsilon}\right]\left(\frac{A_2^v}{\epsilon^2} + \frac{A_1^v}{\epsilon} + A_0^v\right)$$

  with

$$A_2^v = -2C_F$$
$$A_1^v = -3C_F$$
$$A_0^v = -2C_F(4 - \pi^2/3)$$

- The full two-body weight is given by the sum $d\sigma_{\rm S} + d\sigma_{\rm V} + 2d\sigma_{\rm HC}^{q\to qg}$. The factor of two occurs since there are two quark legs, either of which can emit a gluon.

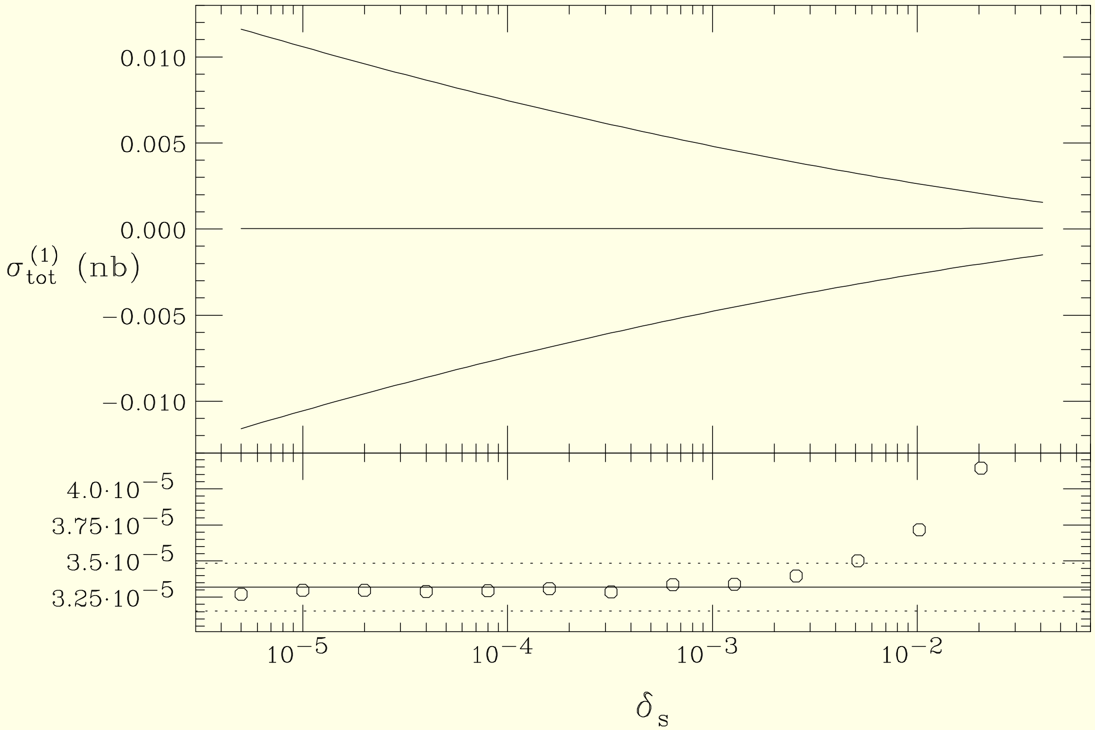
- At this point we have a finite result since $A_2^s + A_2^v$ and $A_1^s + A_1^v + 2A_1^{q\rightarrow qg}$ both separately add up to zero
- The finite two-body weight is given by

$$\sigma^{(2)} = \int d\sigma_0 \left(\frac{\alpha_s}{2\pi}\right)\left(A_0^s + A_0^v + 2A_0^{q\rightarrow qg}\right)$$

while the three-body contribution is given by

$$\sigma^{(3)} = \sigma_{H\overline{C}} = \frac{1}{2s_{12}} \int_{H\overline{C}} \overline{\sum} |M_3|^2 dPS^3$$

- The final result is shown in the following figure as a positive three-body weight, a negative two-body weight and the finite sum



- The results are plotted versus δ_s with $\delta_c = \delta_s/300$
- The solid horizontal line is the exact result
- The method converges nicely, provided that the cut-offs are small enough
- Note: the small triangular regions denoted by m in the phase space figure are not included in the calculation. Their contribution can be included, but it is of order δ_c/δ_s and is negligible provided that $\delta_c << \delta_s$