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Fragmentation Functions of Light and Heavy Quarks

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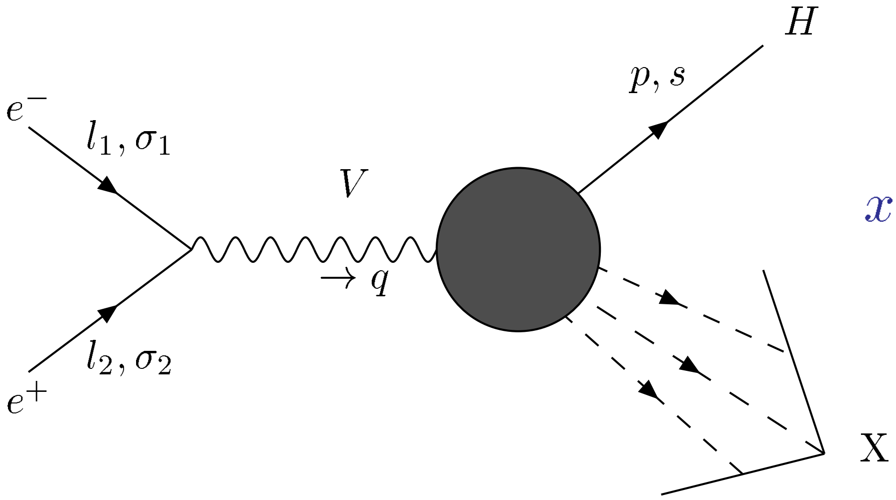
Outline

- Definitions and evolution:
  similarities and differences with structure functions
- Light flavours FF
- Heavy flavours FF
  - differences with light flavours case and various approaches: phenomenological, pQCD, etc.
  - Heavy flavours FF as a theoretical tool for improving pQCD predictions
  - Fragmentation functions for heavy quarkonia

Definition of Fragmentation Functions



$$x \equiv x_E = \frac{2E}{Q} = \frac{2p \cdot q}{q^2}$$

The experimentally observable fragmentation function is the energy fraction distribution $\frac{1}{\sigma_{tot}} d\sigma^H/dx$ for the hadron H in the process $e^+e^- \to \gamma, Z \to H + X$.

Via the QCD factorization theorem this can be written as a convolution of short distance coefficient functions C_i and decay functions D_i^H:

$$\frac{d\sigma^H}{dx} = \sum_i \int_x^1 \frac{dz}{z} C_i(z, \alpha_s(\mu), Q, \mu) D_i^H(x/z, \mu)$$

- $C_i(z, \alpha_s(\mu), Q, \mu)$: production of parton i in the short distance interaction. Process dependent.
- $D_i^H(x/z, \mu)$: fragmentation of the parton i into the hadron H. Universal.
- μ: factorization scale. Separates short- and long-distance physics

In general, fragmentation functions $\neq$ decay functions, in the same way as structure functions $\neq$ parton distribution functions. The former are experimentally measurable cross sections, the latter scheme dependent theoretical tools.

However, in zeroth order QCD (= parton model) the coefficient functions are simply the cross sections for $e^+e^- \to \gamma, Z \to q\bar{q}$:

$$C_i(z,Q) = \frac{4\pi\alpha_{em}^2 e_q^2}{Q^2}\delta(1-z) \qquad (\gamma\text{−exchange only here})$$

Therefore

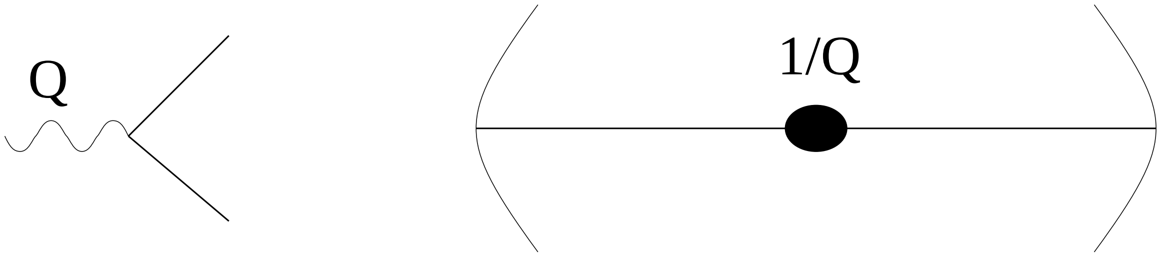
$$\begin{aligned} \frac{d\sigma^H}{dx} &= \frac{4\pi\alpha_{em}^2}{Q^2}\sum_i e_i^2 \int_x^1 \frac{dz}{z}\delta(1-z)D_i^H(x/z) = \\ &= \frac{4\pi\alpha_{em}^2}{Q^2}\sum_i e_i^2 D_i^H(x) = \sum_i \sigma_i^0 D_i^H(x) \end{aligned}$$

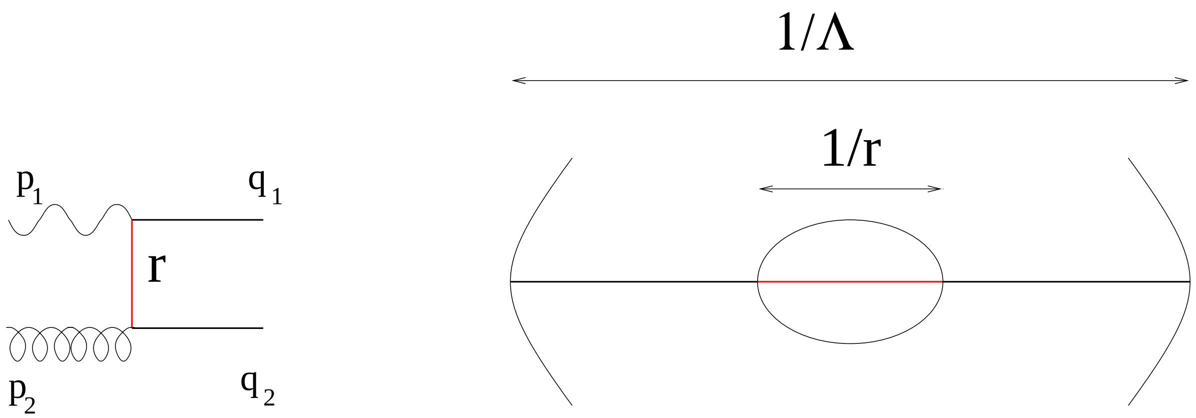
Hence the abused identity between fragmentation and decay functions. This is the time-like analogue of what happens in DIS definition of structure functions.

Note also the absence, at this order, of the factorization scale μ.

Final note: we shall keep abusing the language in these lectures, and always talk about fragmentation functions...:-(

Scale of the FF





In e^+e^- collisions (top) the size of the hard interaction is given by $1/Q = 1/\sqrt{s}$, with $\sqrt{s}$ center-of-mass energy. Hence the FF scale is taken to be Q.

In, say, photon-gluon interactions (bottom) the size of the interaction is not directly related to $\sqrt{s}$. It is rather given by $1/r$, with $r = p_1 - q_1$, i.e. the momentum transfer.

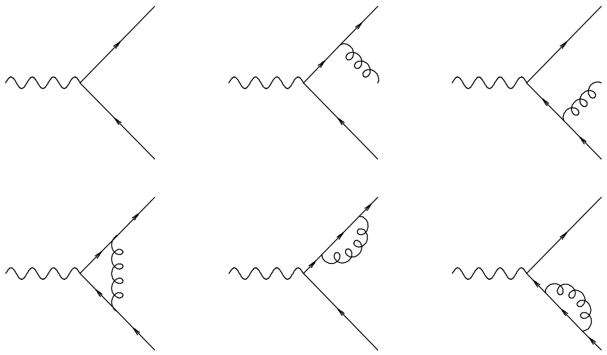
Using $p_1 = (E, 0, E)$ and $q_1 = (E, p_T, \sqrt{E^2 - {p_T}^2})$ we have

$$r^2 = \left(0, p_T, E - \sqrt{E^2 - {p_T}^2}\right)^2 = -2E^2 + 2E\sqrt{E^2 - {p_T}^2}$$

Hence, if $p_T \to 0$ also $r \to 0$, regardless of $\sqrt{s}$. p_T is in this case the hard scale.

Higher order QCD corrections

Next-to-leading order QCD corrections can be calculated by evaluating the following diagrams:



- born + virtual corrections for $V \to q\bar{q}$
- real gluon emission $V \to q\bar{q}g$

The arising of collinear divergences calls for the factoring out of the singularities. The resulting coefficient functions in the $\overline{\mathrm{MS}}$ scheme are:

$$
\begin{aligned}
C_q(z,\mu/Q) &\propto \delta(1-z)+\frac{\alpha_s(Q)C_F}{2\pi}\left[\log\frac{Q^2}{\mu^2}\left(\frac{1+z^2}{1-z}\right)_+ +2\frac{1+z^2}{1-z}\log z\right.\\
&+ \left(\frac{\log(1-z)}{1-z}\right)_+(1+z^2)-\frac{3}{4}\left(\frac{1}{1-z}\right)_+(1+z^2)\\
&+ \left.\frac{9+5z}{4}-\frac{7}{2}(1+z)+\left(\frac{2}{3}\pi^2-\frac{9}{2}\right)\delta(1-z)\right]\\
C_g(z,\mu/Q) &\propto 2\frac{1+(1-z)^2}{z}\left[\log\frac{Q^2}{\mu^2}+\log(1-z)+2\log z\right]
\end{aligned}
$$

Note the non trivial μ–dependence, remnant of the factorization procedure. It will be compensated in the observable cross section by the proper dependence of the evolved fragmentation (well, decay....) functions.

Factorizations schemes different from $\overline{\mathrm{MS}}$ can of course be defined, e.g. the annihilation scheme.

Altarelli-Parisi evolution for fragmentation functions

In a way identical to the parton distribution functions, also the fragmentation functions obey QCD evolution equations. However, the equations themselves are NOT identical.

In leading order QCD the set of equations is simply the transpose of the PDF one:

$$\frac{\partial D_q(x,\mu)}{\partial \log \mu^2} = \frac{\alpha_s(\mu)}{2\pi}\left[P^{(0)}_{qq} \otimes D_q(\mu) + P^{(0)}_{gq} \otimes D_g(\mu)\right]$$

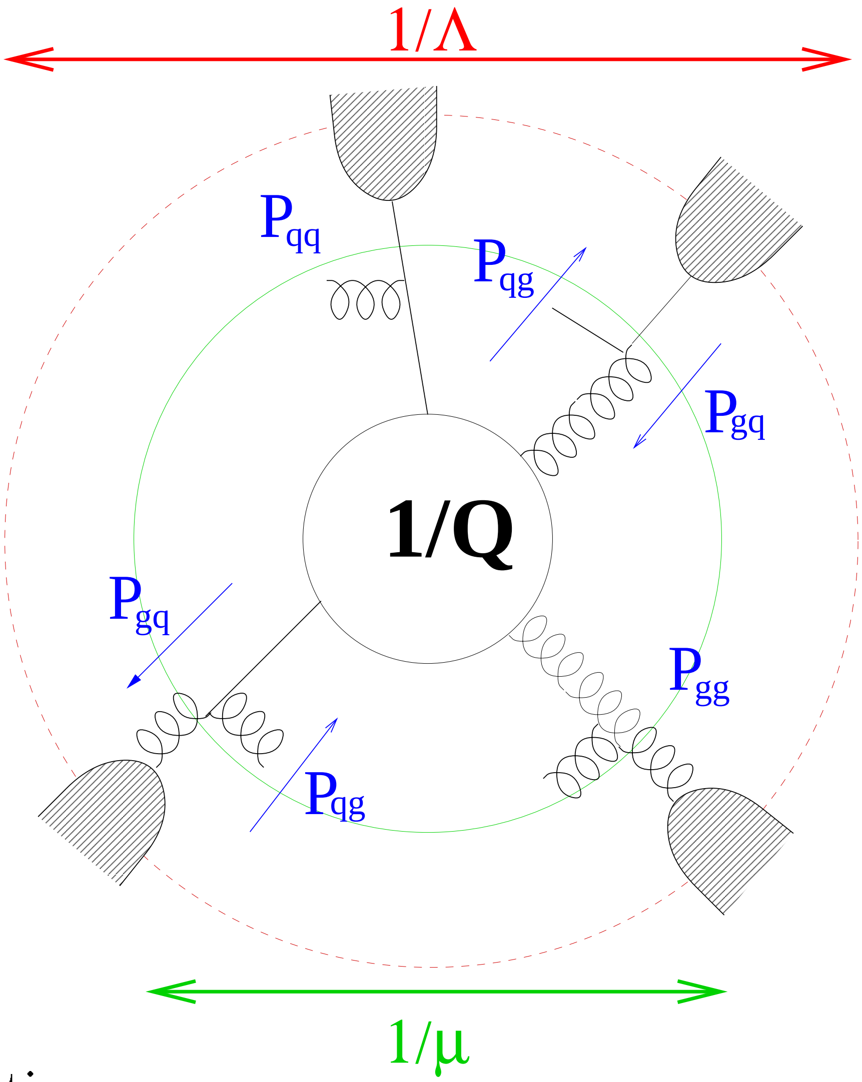
$$\frac{\partial D_g(x,\mu)}{\partial \log \mu^2} = \frac{\alpha_s(\mu)}{2\pi}\left[P^{(0)}_{qg} \otimes \sum_{q,\bar{q}} D_q(\mu) + P^{(0)}_{gg} \otimes D_g(\mu)\right]$$

with

$$P^{(0)}_{gq} = C_F \frac{1+(1-z)^2}{z} \qquad\qquad P^{(0)}_{qg} = T_F\left[z^2 + (1-z)^2\right]$$

$$q \to g + q \qquad\qquad g \to q + \bar{q}$$

In next-to-leading order also the form of the AP splitting vertices changes, because the splitting processes in the time-like and in the space-like regions differ.



Structure functions:

$$F_q(Q) - F_q(\mu) \sim \frac{\alpha_s}{2\pi} \log \frac{Q^2}{\mu^2} \left[F_q P_{qq} + F_g P_{qg} \right]$$

$$F_g(Q) - F_g(\mu) \sim \frac{\alpha_s}{2\pi} \log \frac{Q^2}{\mu^2} \left[F_g P_{gg} + F_q P_{gq} \right]$$

Fragmentation functions:

$$D_q(Q) - D_q(\mu) \sim \frac{\alpha_s}{2\pi} \log \frac{Q^2}{\mu^2} \left[P_{qq} D_q + P_{gq} D_g \right]$$

$$D_g(Q) - D_g(\mu) \sim \frac{\alpha_s}{2\pi} \log \frac{Q^2}{\mu^2} \left[P_{gg} D_g + P_{qg} D_q \right]$$

Fragmentation functions for light hadrons

Light hadrons (π, K, ϕ, η, Λ, p, etc.) are copiously produced in e^+e^- collisions.

Partons produced in the hard interaction (quarks and gluons) are said to fragment into these observed hadronic final states.

Being the masses of the constituent quarks smaller than the Λ_{QCD} scale, their formation involves essentially non perturbative dynamics.

Hence, perturbative QCD cannot predict the details of these formation processes. Phenomenological models can of course be implemented in Montecarlo programs (string models, cluster models, etc.), but we will not describe them here.

Thanks to the factorization theorem one can however fit universal (but scheme dependent) fragmentation functions to data and use them with predictive power in different reactions. This is conceptually identical to what is done when fitting parton distribution functions.

Sketch of the methodology for light quarks fragmentation functions

As a starting point, one chooses a flexible enough functional form for the fragmentation functions at an initial scale μ_0 of the order of a few GeV, e.g.

$$D_i^H(z, \mu_0) = N_i \; z^{\alpha_i} (1-z)^{\beta_i}$$

Advantage of this form → simple Mellin transform:

$$\begin{aligned} D_N(\alpha, \beta) &\equiv \int_0^1 dz \; z^{N-1} D(z; \alpha, \beta) \\ &\propto \frac{\Gamma(\alpha+N)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta+N)} \end{aligned}$$

These initial conditions are evolved via the Altarelli-Parisi equations up to the energy scale of the experiment, and convoluted with the proper coefficient functions. A cross section is therefore obtained and the free parameters $\{N_i$, α_i, $\beta_i\}$ are fitted to the data.

A hadron-hadron cross section can then for instance be obtained using

$$d\sigma \sim \sum_{i,j,k} \int F_i F_j d\hat{\sigma}(ij \to kX) D_k^H$$

Example: fit to e^+e^- data

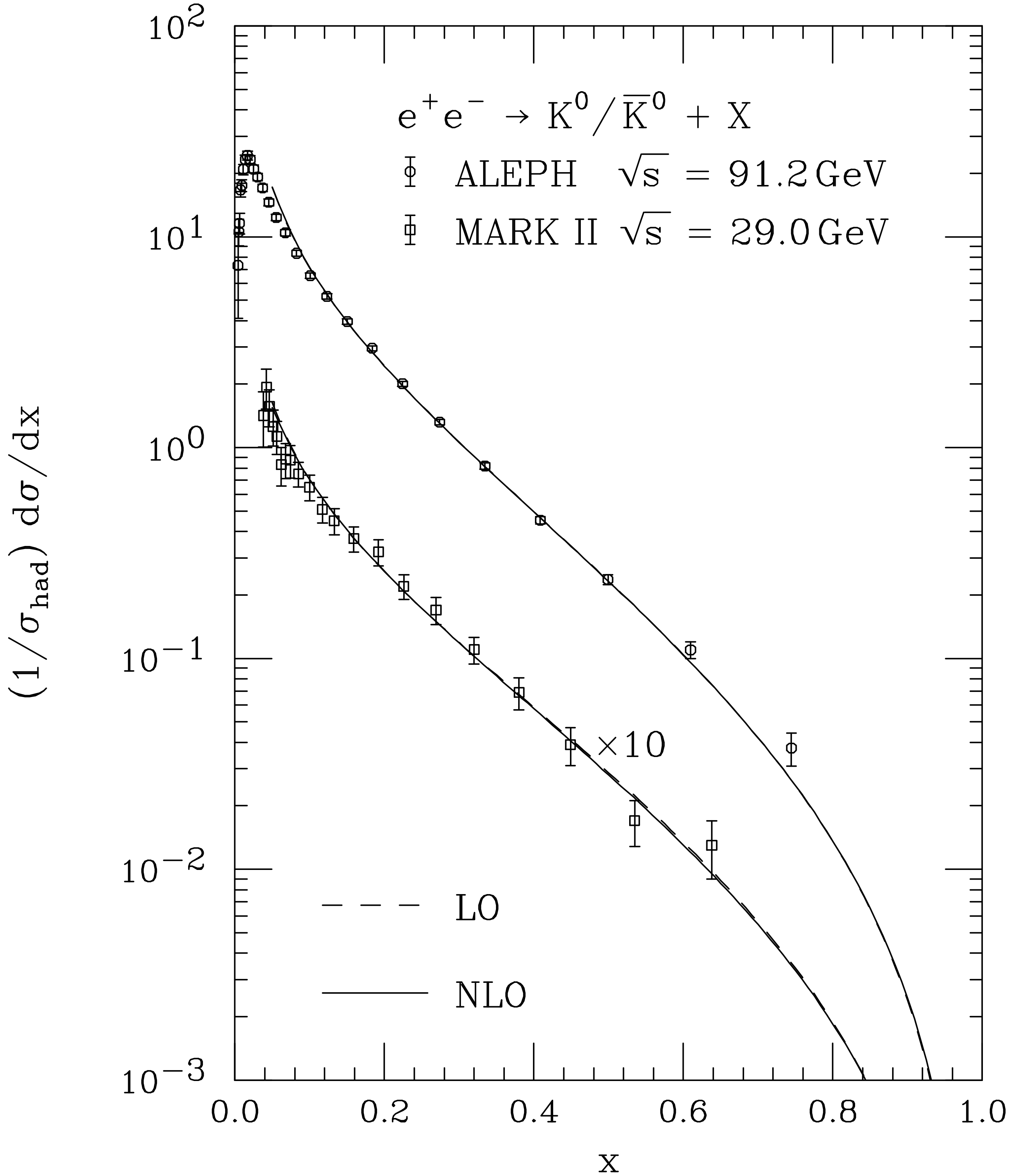


Fig. 2

Example of scaling violations in e^+e^- data

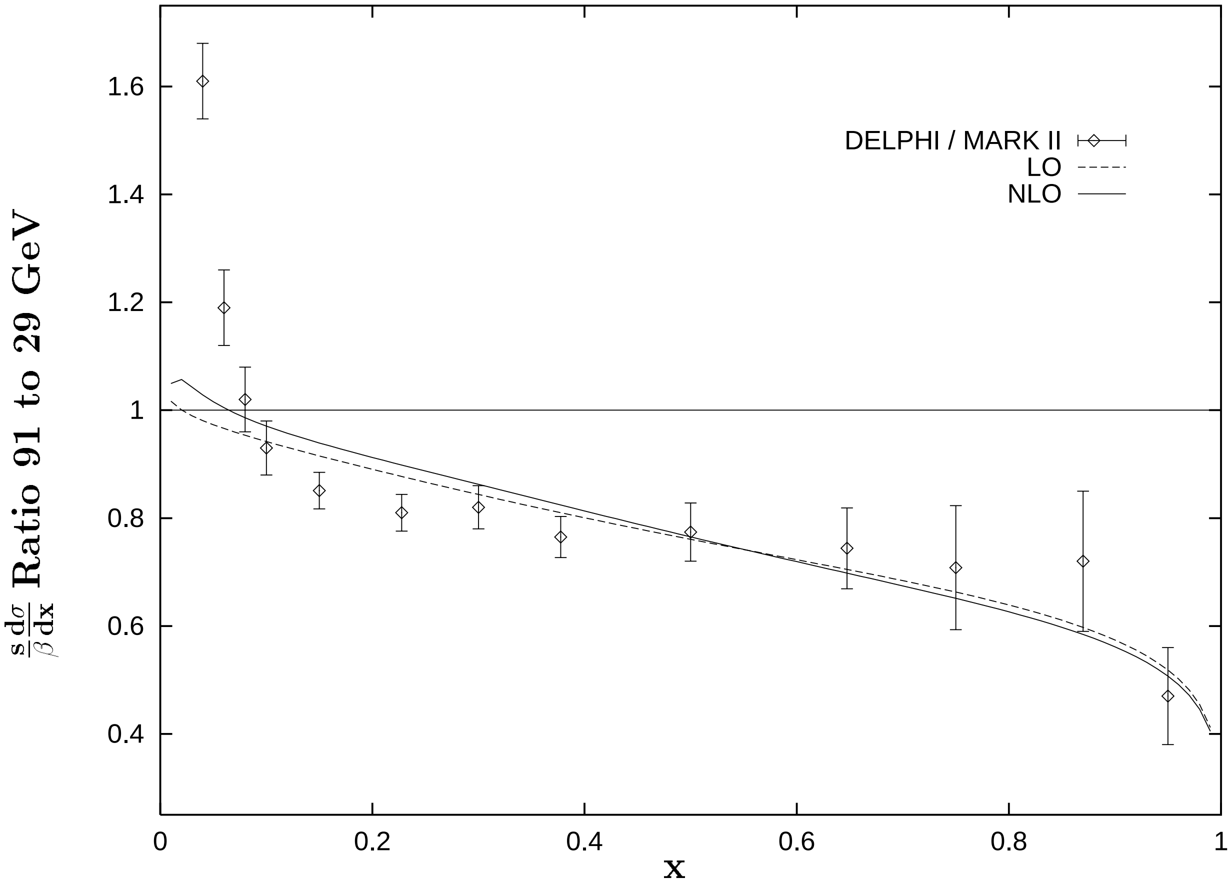


Fig. 9

Allows a determination of Λ_{QCD} and hence of α_s.

Example: prediction for $p\bar{p}$ data

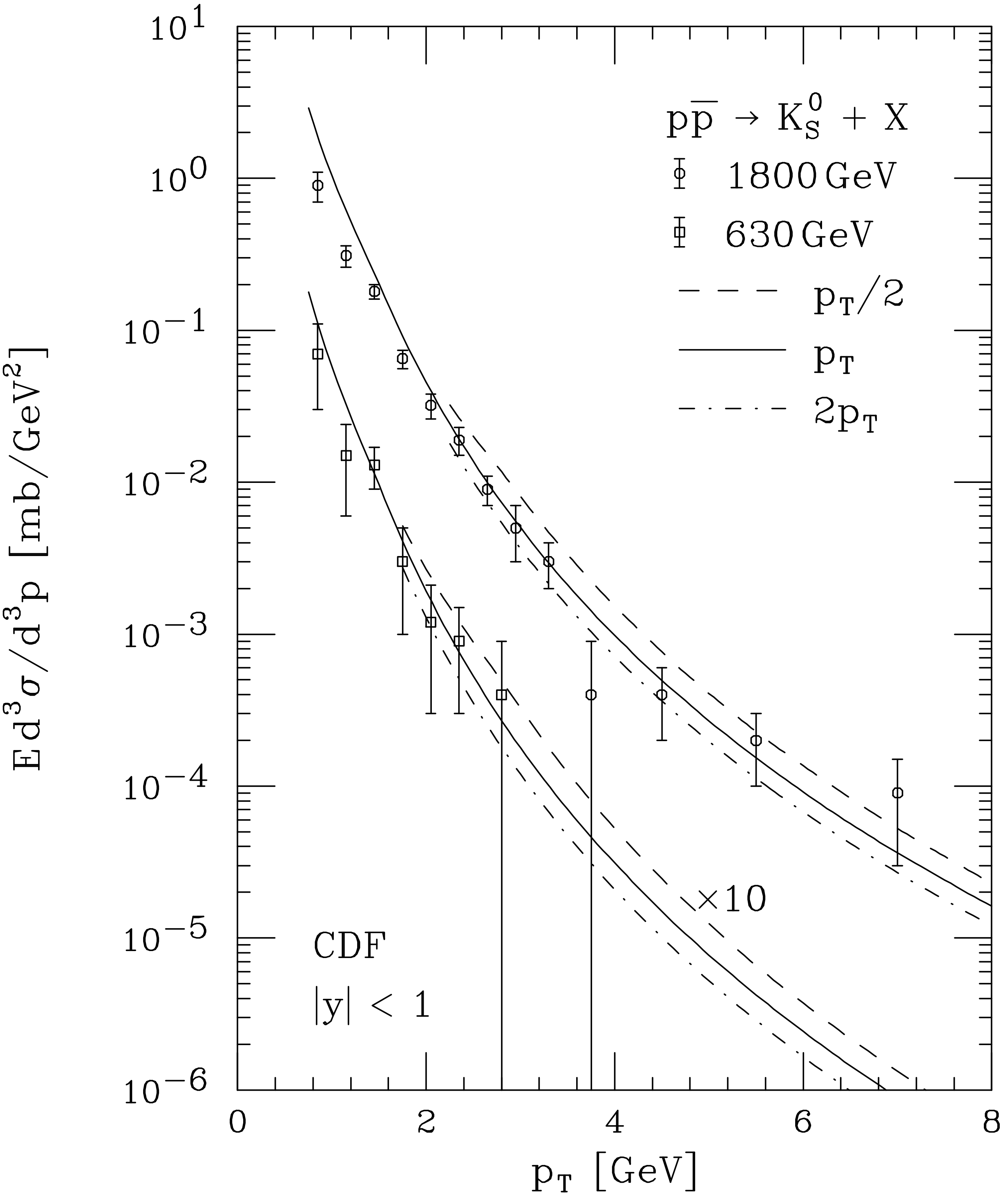


Fig. 5

Fragmentation functions for heavy quarks

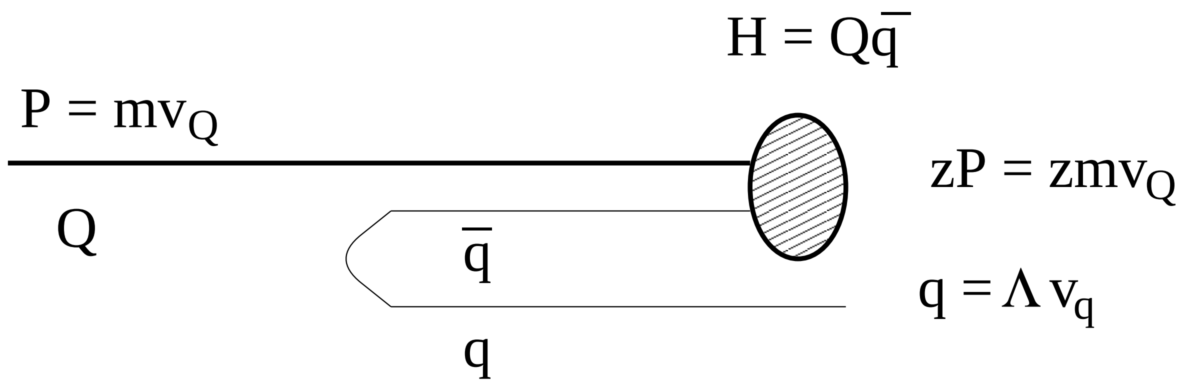
The heavy quark (Q) mass $m > \Lambda_{QCD}$ provides a hard, perturbative scale. The fragmentation functions can therefore be analyzed to a greater extent within perturbative QCD.

Main phenomenological difference with light quarks fragmentation: the FF are peaked close to $x = 1$ rather that close to $x = 0$: a heavy quark Q is only marginally slowed down when picking up a light quark to form a heavy meson.

Various approaches are possible:

- fully phenomenological models, like the Peterson et al. FF
- perturbative QCD only approaches
- HQET approaches
- perturbative QCD + non perturbative parametrizations

Bjorken's argument



The heavy quark has momentum $P = mv_Q$, the light quarks have momentum $q = \Lambda v_q$, width Λ a hadronic mass scale

For the binding we need $v_Q \simeq v_q = v$

We have then

$$\begin{aligned} P &= zP + q \\ mv &= zmv + \Lambda v \end{aligned}$$

and therefore

$$z \simeq 1 - \frac{\Lambda}{m}$$

Hence, the FF is peaked near $z = 1$ and binding effects scale linearly in the heavy quark mass (NB. more rigorous derivations of this result are available)

HQET predictions for heavy-light mesons

Fragmentation functions for $h \to H(P)$ can be defined in terms of matrix elements. HQET can then be used to organize them in series of $1/M$:

$$
\begin{aligned}
D_h^H(1/x) \quad &\equiv \quad \frac{1}{4\pi} \sum_{\chi} \int d\lambda e^{i\lambda x} \\
\times \quad & \langle 0 | n \cdot \gamma_{ij} h_j(\lambda n) | H(P)\chi \rangle \langle H(P)\chi | \bar{h}_i(0) | 0 \rangle
\end{aligned}
$$

The outcome of the analysis is that the leading and next-to-leading terms in $\epsilon = (M-m)/M$ behave as

$$
D_h^H(z) = \frac{1}{\epsilon} a \left[\frac{1}{\epsilon} \left(\frac{1}{z} - \frac{m}{M} \right) \right] + b \left[\frac{1}{\epsilon} \left(\frac{1}{z} - \frac{m}{M} \right) \right]
$$

with $a(y)$ and $b(y)$ being functions containing non perturbative parameters, but no mass dependence.

The form of the result means that the FF shrinks towards $z = 1$ linearly in $1/M$ (recall Bjorken's argument).

Phenomenological fragmentation models must therefore contain this behaviour to be consistent with QCD.

Phenomenological models

Description of the transition $Q \to H(Q\bar{q}) + q$, where Q has momentum P and H has momentum zP. Model due to Peterson, Schlatter, Schmitt, Zerwas.

Amplitude$(Q \to H + q) \propto \Delta E^{-1} = E_H + E_q - E_Q$

$$\begin{aligned}\Delta E &= (m_Q^2 + z^2 P^2)^{1/2} + (m_q^2 + (1-z)^2 P^2)^{1/2} \\ &- (m_Q^2 + P^2)^{1/2} \\ &\propto 1 - \frac{1}{z} - \frac{\epsilon_Q}{1-z}\end{aligned}$$

Squaring and including a factor z^{-1} for longitudinal phase space we get

$$D_Q^H(z) \propto \frac{1}{z\left[1 - \frac{1}{z} - \frac{\epsilon_Q}{1-z}\right]^2}$$

with $\epsilon_Q \sim m_q^2/m_Q^2$.

For small ϵ_Q we have $\langle z \rangle \simeq 1 - \mathcal{O}(\sqrt{\epsilon_Q})$.

Perturbative QCD

A calculation of $d\sigma^Q/dx$ for the process $e^+e^- \to \gamma, Z \to Q + X$ is possible without the need to factor out singularities:

$$
\begin{aligned}
\frac{d\sigma^Q}{dx} &= \delta(1-x) + \frac{\alpha_s(Q^2)}{2\pi}\left\{C_F + C_F\left[\log\frac{Q^2}{m^2}\left(\frac{1+x^2}{1-x}\right)_+ + 2\frac{1+x^2}{1-x}\log x\right.\right. \\
&- \left(\frac{\log(1-x)}{1-x}\right)_+(1+x^2) + \frac{1}{2}\left(\frac{1}{1-x}\right)_+(x^2-6x-2) \\
&+ \left.\left.\left(\frac{2}{3}\pi^2 - \frac{5}{2}\right)\delta(1-x)\right]\right\} + \mathcal{O}\left(\frac{m}{Q}\right)
\end{aligned}
$$

The logarithmic term $\log(Q^2/m^2)$ can however grow large and spoil the perturbative series. This formula can therefore be improved using the fragmentation functions language. We rewrite it in the form:

$$
\begin{aligned}
\frac{d\sigma^Q}{dx} &= C_i^{\overline{\mathrm{MS}}}(Q,\mu) \otimes D_i^Q(\mu, m) \\
&= C_i^{\overline{\mathrm{MS}}}(Q,\mu) \otimes E_{ik}(\mu,\ \mu_0) \otimes D_k^Q(\mu_0, m)
\end{aligned}
$$

The initial condition $D_i^Q(\mu_0, m)$ can be calculated in perturbative QCD, thanks to the large heavy quark mass. Hence, no need to fit data at this stage.

The evolution factor $E_{ik}(\mu,\ \mu_0)$, from the Altarelli-Parisi equations, resum the large logs, improving the fixed order formula.

Perturbative Fragmentation Functions

Comparing the $\overline{\mathrm{MS}}$ coefficient functions and the NLO massive calculation one can extract initial conditions for the $D_i^Q(x, \mu_0)$ FF (Mele, Nason):

$$D_Q^Q(x,\mu_0,m)=\delta(1-x)+\frac{\alpha_s(\mu_0)C_F}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu_0^2}{m^2}-2\log(1-x)-1\right)\right]_+$$

$$D_g^Q(x,\mu_0,m)=\frac{\alpha_s(\mu_0)T_F}{2\pi}(x^2+(1-x)^2)\log\frac{\mu_0^2}{m^2}$$

$$D_{q,\bar{q},\bar{Q}}^Q(x,\mu_0,m)=0$$

In Mellin moments space, the NLO AP evolution equation for the $Q \to Q$ component only is given by:

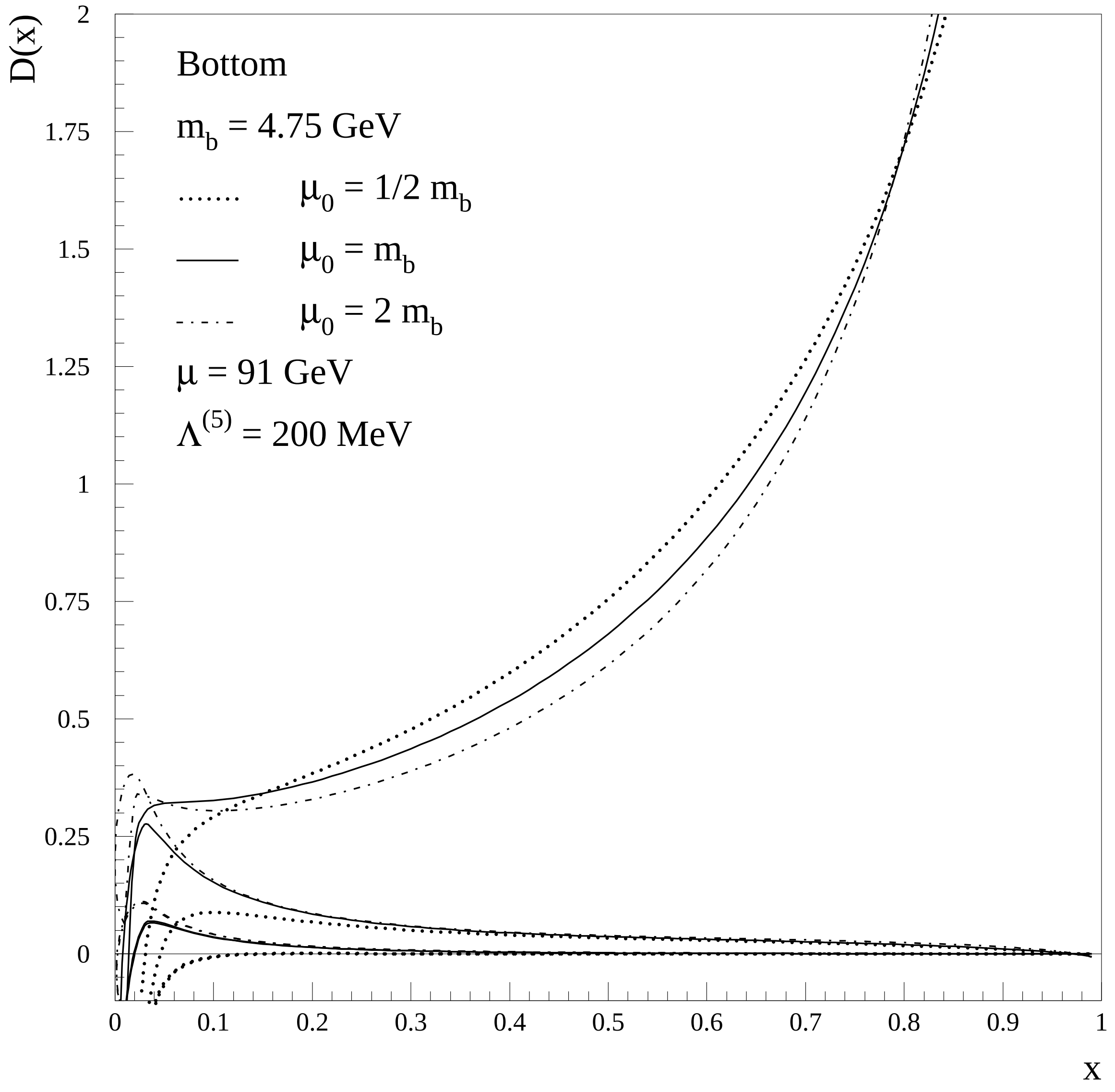
$$\frac{dD_N(\mu)}{d\log\mu^2} = \gamma_N^{(2)} D_N(\mu)$$

$$\gamma_N^{(2)} = \frac{\alpha_s(\mu)}{2\pi}\left[P_N^{(0)} + \frac{\alpha_s(\mu)}{2\pi}P_N^{(1)}\right]$$

Its analytical solution reads

$$D_N(\mu)=D_N(\mu_0)\left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right]^{\frac{P_N^{(0)}}{2\pi b_0}}\exp\left\{\frac{(\alpha_s(\mu_0)-\alpha_s(\mu))}{4\pi^2 b_0}\left(P_N^{(1)}-\frac{2\pi b_1}{b_0}P_N^{(0)}\right)\right\}$$

Example: bottom fragmentation function

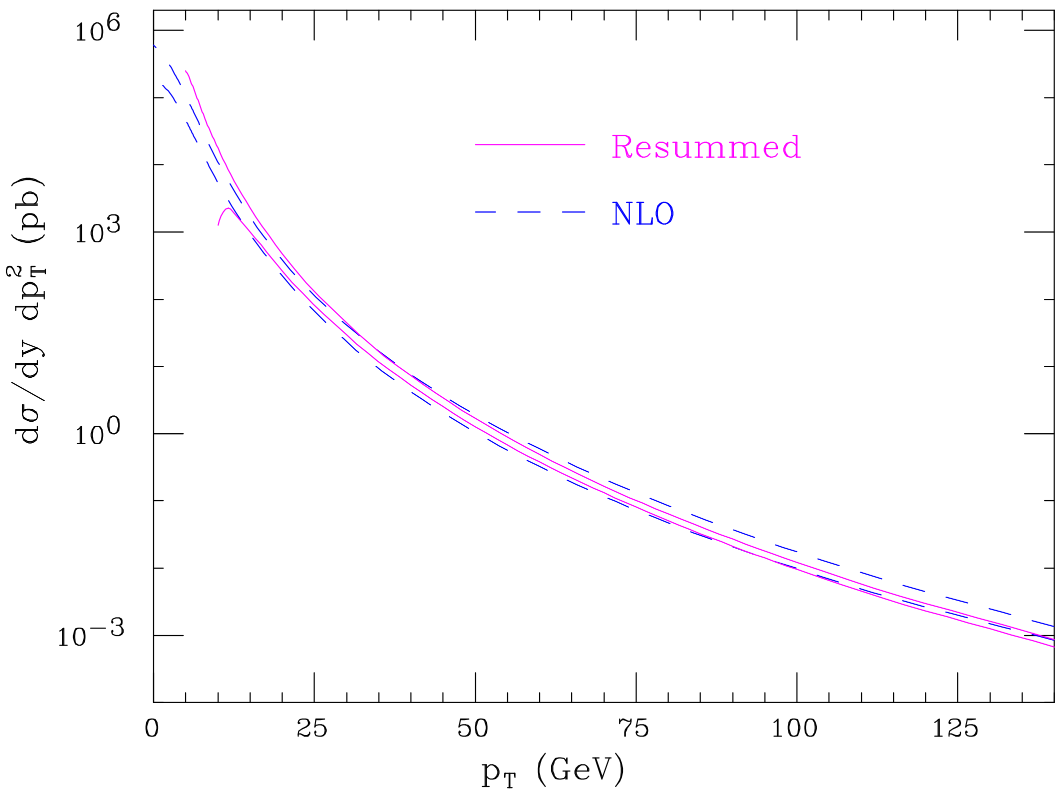


Perturbative Fragmentation Functions

The evolved $D_i^Q(x,\mu,m)$ perturbative fragmentation functions for heavy quarks are, according to the factorization theorem, a universal (though scheme dependent) object.

Their are not restricted to e^+e- collisions, but can also be employed in other reactions, like the light flavours fragmentation functions.

They become therefore a theoretical tool for resumming large collinear logs where required, e.g. in large-p_T hadron-hadron or photon-hadron collisions. This resummation results in a milder factorization scale dependence and therefore in a more reliable theoretical prediction:



Resummation of $\log(p_T/m)$ terms

$\alpha_s \log(p_T/m)$ terms are generated by gluon splitting to $Q\overline{Q}$ or by gluon emission from a heavy quark. They do not affect total cross sections but grow large in the large p_T region.

In the same region, m/p_T terms become small.

$\Rightarrow$ resummation technique: use massless quarks cross section convoluted with parton distribution functions $F_{Q/H}$ and fragmentation functions D_k^Q to describe the quasi-singular splittings:

$$d\sigma(H_1 H_2 \to QX) = \sum_{ijk} \int F_{i/H_1} F_{j/H_2} d\hat{\sigma}(ij \to k) D_k^Q$$

with i, j, k also equal to the heavy quark Q.

Initial state conditions for $F_{Q/H}(\mu = m)$ [Collins, Tung] and $D_i^Q(\mu = m)$ [Mele, Nason] can be calculated in pQCD at NLO level in the $\overline{\text{MS}}$ scheme:

$$F_{Q/H}(x,\mu_0=m)=0$$

$$D_Q^Q(x,\mu_0)=\delta(1-x)+\frac{\alpha_s(\mu_0)C_F}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu_0^2}{m_Q^2}-2\log(1-x)-1\right)\right]_+$$

$$D_g^Q(x,\mu_0)=\frac{\alpha_s(\mu_0)T_F}{2\pi}(x^2+(1-x)^2)\log\frac{\mu_0^2}{m_Q^2}$$

$$D_{q,\bar{q},\bar{Q}}^Q(x,\mu_0)=0$$

$\to$ Evolution up to $\mu = p_T$ resums the large logs.

Perturbative QCD
+
non perturbative parametrization

Fit of a non-perturbative $Q \to$ heavy-meson fragmentation function (e.g. $c \to D^*$) in e^+e^- data

We describe heavy mesons fragmentation function in e^+e^- by [Mele, Nason; Colangelo, Nason]:

$$\frac{d\sigma^H}{dx} = \sum_{i=q,g,Q} \int_x^1 \frac{dz}{z} C_i(z, \alpha_s(\mu), Q, \mu) D_i^H \left(\frac{x}{z}, \mu\right)$$

with

$$D_i^H(\mu) = D_i^Q(\mu) \otimes D_{np}(\epsilon)$$

$D_{np}(\epsilon)$ can be the Peterson form:

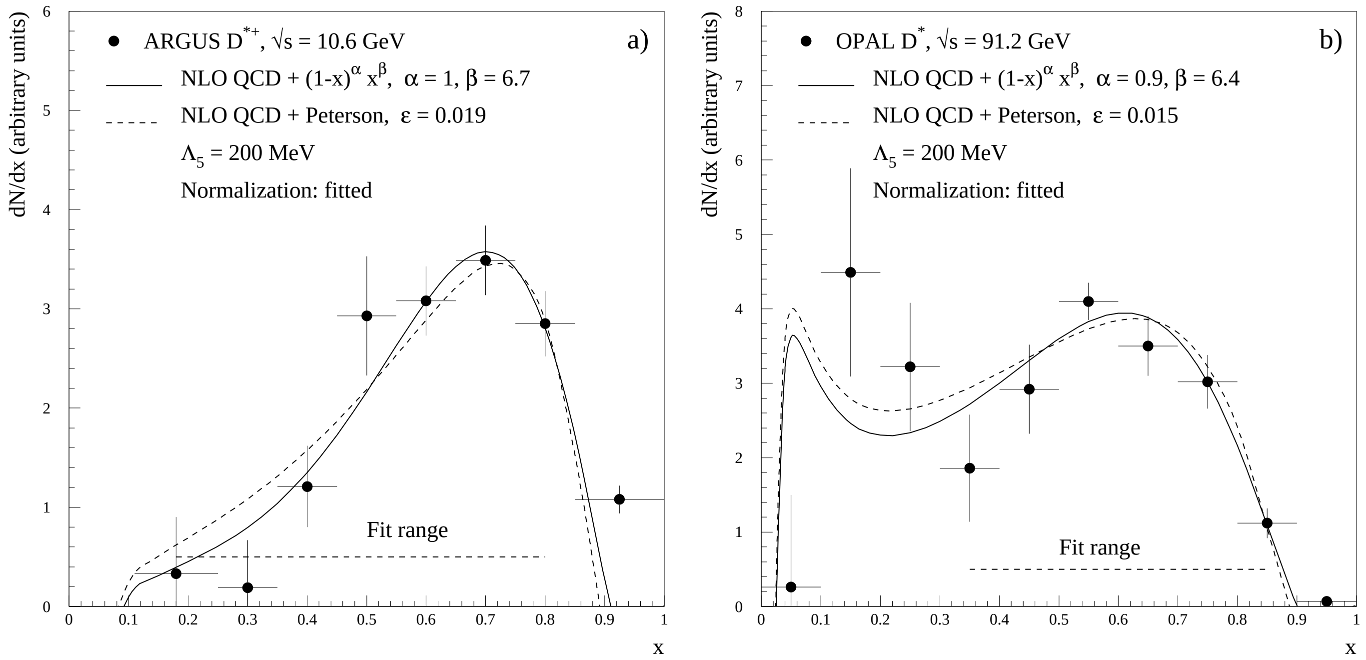
$$D_{np}(\epsilon) \sim \frac{1}{x\left[1 - 1/x - \epsilon/(1-x)\right]^2}$$

or another non-perturbative parametrization

Hypothesis: all scaling violations accounted for by the evolution of the perturbative component of the fragmentation function, i.e. $D_i^Q(\mu)$

$\Rightarrow$ the same ϵ should describe data at different center-of-mass energies. Check: ARGUS (10.6 GeV) and OPAL (91.2 GeV) data on D^* production

Fits to $D_c^{D^*}(\epsilon)$ in e^+e^-



	Λ_5 = 100 MeV	Λ_5 = 200 MeV	Λ_5 = 300 MeV
		Next-to-leading log evolution	
ARGUS	.031	.019	.011 ± .003
OPAL	.033 ± .005	.015 ± .002	.008 ± .001
		Leading log evolution	
ARGUS	.07	.055	.036
OPAL	.10 ± .01	.08 ± .01	.06 ± .01

- ϵ stable with scale (with NLL evolution)
- $\epsilon_{\rm NLL} \sim 0.02 < \epsilon_{\rm LL} \sim 0.06$ ($\Rightarrow$ harder non-pert. fragm. funct.)

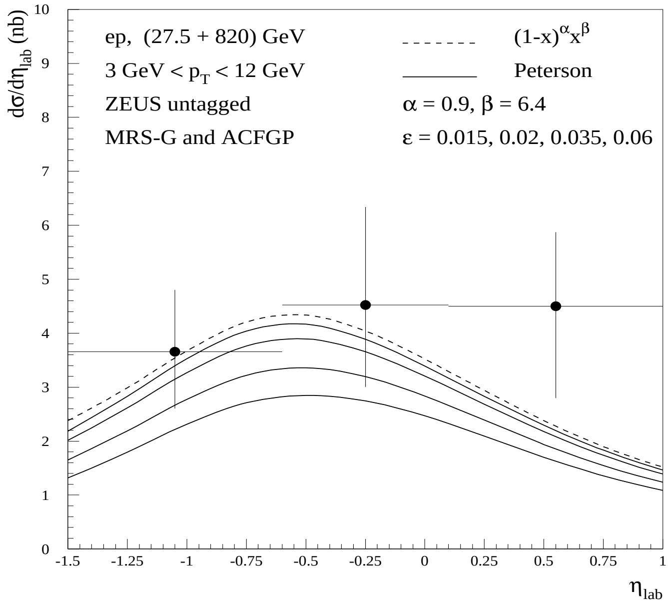
because:

- LL evolution insufficient to account for scaling violations
- NLL evolution softens more the perturbative part of the FF

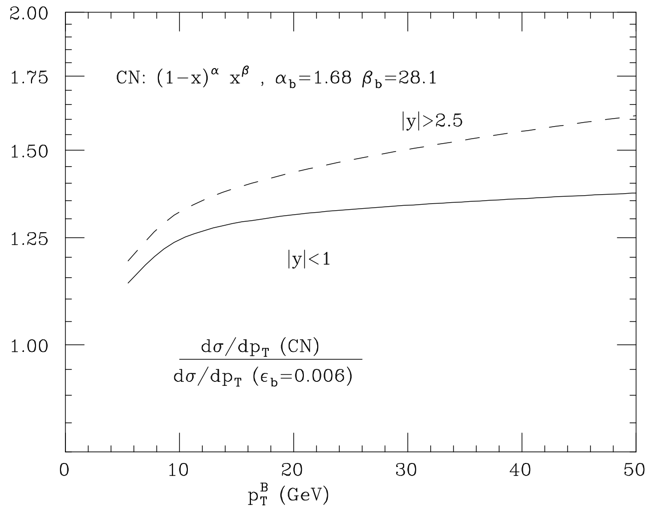
$\Rightarrow$ the non-pert. parameter must match the underlying perturbative description. So far, even NLO calculation have been complemented with $\epsilon_c \sim 0.06$, as extracted by LL Montecarlo simulations [Chrin]

Effect of a smaller ϵ on photon-hadron and hadron-hadron NLO cross sections:

D^* at HERA:



B at the Tevatron [Mangano]:



Relative increase of B cross section, using non-pert. parameters from fits to B e^+e^- data [Colangelo, Nason], roughly equivalent to

$$\epsilon_b \sim 0.002 \leftarrow \epsilon_c \frac{m_c^2}{m_b^2}$$

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