Partial summations of perturbation theory

D. E. Soper

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A familiar example

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

- $Q^2 = (\text{c.m. energy})^2$
- $M =$ a very big mass $\gg Q$
  - theory "defined" at momentum scale M
  - $\alpha_s(M)$: coupling at this "fundamental" scale.

$$R = R_0 \left\{ 1 + \frac{1}{\pi}\,\alpha_s(M^2) + \left[\frac{\beta_0}{\pi}\ln\left(\frac{M^2}{Q^2}\right) + C_{2,0}\right]\alpha_s(M^2)^2 + \left[\frac{\beta_0^2}{\pi}\ln^2\left(\frac{M^2}{Q^2}\right) + C_{3,1}\ln\left(\frac{M^2}{Q^2}\right) + C_{3,0}\right]\alpha_s(M^2)^3 + \cdots \right\}.$$

- $\alpha_s \ll 1$ but $\ln(M^2/Q^2) \gg 1$

$\Rightarrow$ the expansion is useless.

Summation to the rescue

Use the renormalization group. Find

$$R = R_0 \left\{ 1 + \frac{1}{\pi}\,\alpha_s(M^2)\,\frac{1}{1-\beta_0 \ln(M^2/Q^2)\alpha_s(M^2)} + f_2\left(\ln(M^2/Q^2)\,\alpha_s(M^2)\right)\,\alpha_s(M^2)^2 + f_3\left(\ln(M^2/Q^2)\,\alpha_s(M^2)\right)\,\alpha_s(M^2)^3 + \cdots \right\}.$$

- The first term sums the "leading logs."
- The next term sums the "next-to-leading logs."
  * To find f_2:
    Use the two loop beta function
    Calculate graphs for R to one more loop.
- Third term sums the "next-to-next-to-leading logs," *etc.*
- This expansion is useful when $\ln(M^2/Q^2)\,\alpha_s(M^2) \lesssim 1$.

Does it agree with experiment?

What we have found is that

$$R(Q^2) = R_0 \left\{ 1 + \frac{1}{\pi}\, \alpha_s(Q^2) + \cdots \right\}.$$

where the running coupling (in the first approximation) is

$$\alpha_s(Q^2) = \frac{\alpha_s(M^2)}{1 - \beta_0 \ln(M^2/Q^2)\alpha_s(M^2)}$$

- Same holds for lots of experimental observables.
- Get $\alpha_s(Q^2)$ from experiments at many scales Q^2.
- See if $\alpha_s(Q^2)$ runs as advertised.

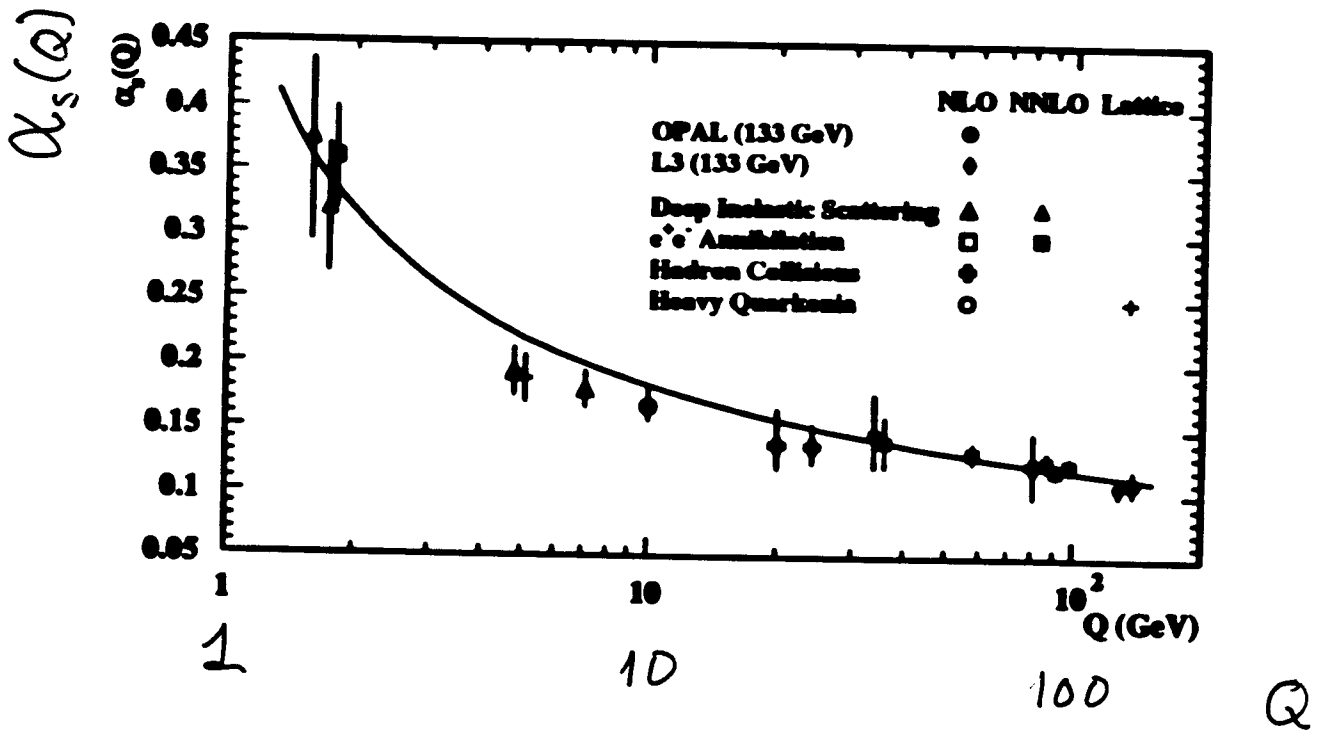


Figure 9: Values of α_s at different energies. The labels NLO and NNLO refer to the order of calculations fitted to extract α_s measurements. NLO corresponds to $O(\alpha_s^2)$ in e^+e^- annihilations, and NNLO to $O(\alpha_s^3)$. The label Lattice refers to α_s values determined from lattice QCD calculations. The curve shows the second order QCD prediction for α_s with $\alpha_s(M_{Z^0})=0.120$.

OPAL Plot

Another familiar example

Consider the DIS structure function F_2.
Neglecting power suppressed corrections,

$$F_2(x,Q^2) \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi,Q^2)\ \hat{F}_2^a(x/\xi,Q^2) + \mathcal{O}(m/Q).$$

- $\hat{F}_2$ is perturbative.
- At lowest order is trivial:

$$\hat{F}_2^a(x/\xi,Q^2) = Q_a^2\ \delta(x/\xi - 1) + \mathcal{O}(\alpha_s),$$

- The $f_{a/h}(\xi,Q^2)$ are parton distribution functions.
- They obey the DGLAP evolution equation

$$\frac{d}{d\log Q} f_{a/h}(x,Q^2) = \sum_b \int_x^1 \frac{d\xi}{\xi}\ P_{ab}(x/\xi,\alpha_s(Q^2))\ f_{b/h}(\xi,Q^2).$$

- Boundary condition: $f_{a/h}(x,Q_0^2)$
- The evolution equation sums a series of the form

$$\sum_n c_n \alpha_s^n(Q^2) \log^n(Q/Q_0)$$

Does it work?

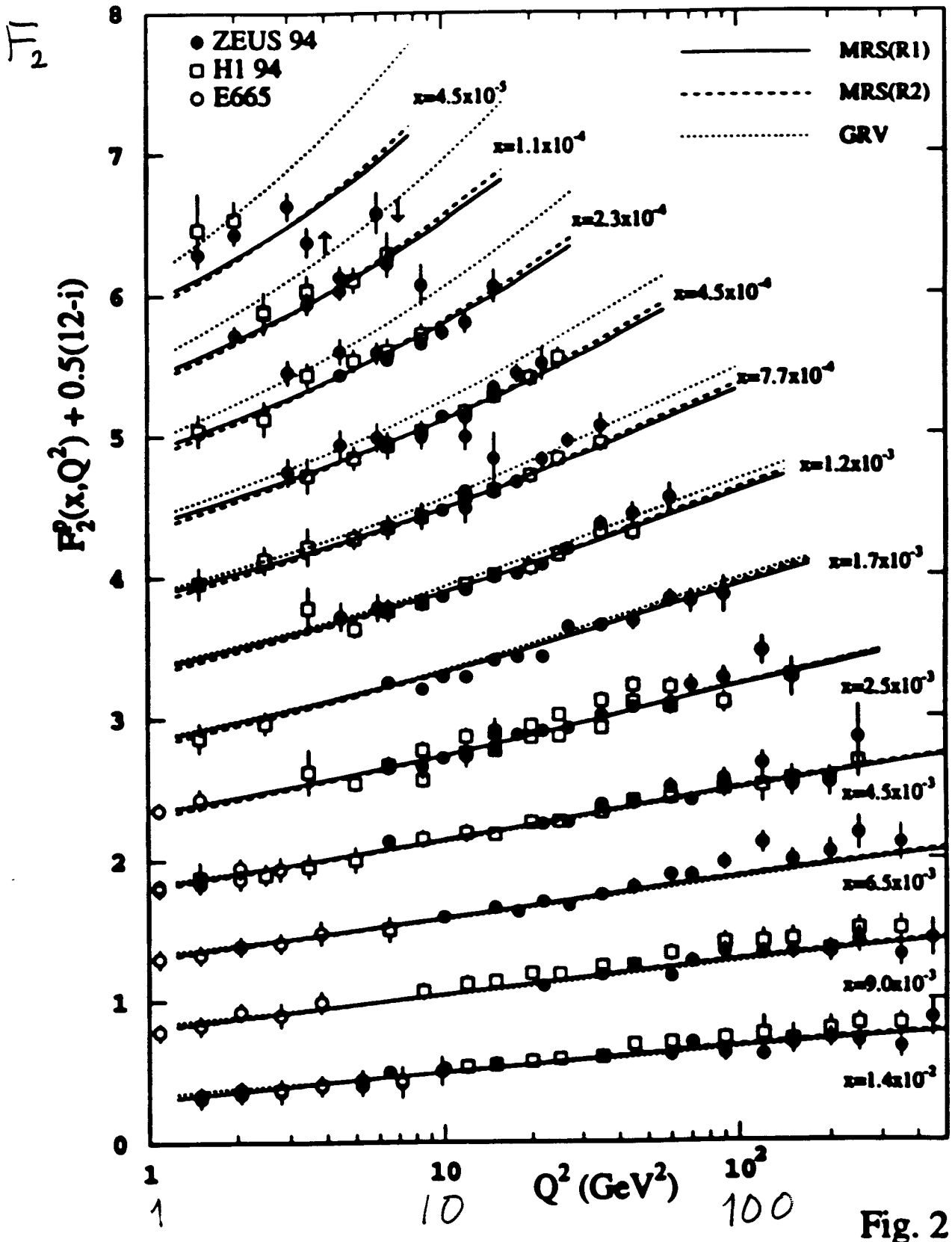


Fig. 2

MRS plot

A harder case

Find the P_T distribution of the Z boson in $p + \bar{p} \to Z + X$ for $P_T \ll M_Z$.
Consider $\hat{F}(P_T)$ defined by

$$\frac{d\sigma}{d^2\mathbf{P}_T\, d\eta} =$$
$$\sum_{ab} \int dx_a\, f_{a/A}(x_A) \int dx_b\, f_{b/B}(x_B)\; \hat{F}_{ab}(P_T, x_A, x_B).$$

With
* Coupling $\alpha_s = \alpha_s(M_Z^2)$
* Large logarithms $L = \log(M_Z^2/P_T^2)$
* Coefficients $c_{nm}(x_A, x_B)$.
* Remainder $Y_n(P_T, x_A, x_B)$, $P_T^2 Y \to 0$ as $P_T \to 0$.

$$\begin{aligned}\hat{F} = \alpha_s &\left[\frac{1}{P_T^2}\left(c_{11}L + c_{10}\right) + Y_1\right] \\ &+ \alpha_s^2 \left[\frac{1}{P_T^2}\left(c_{23}L^3 + c_{22}L^3 + c_{21}L + c_{10}\right) + Y_2\right] \\ &+ \alpha_s^3 \left[\frac{1}{P_T^2}\left(c_{35}L^5 + c_{34}L^4 + \cdots + c_{30}\right) + Y_3\right] \\ &+ \cdots\end{aligned}$$

- Two logs per α_s ("Sudakov").
- Want a summation that is useful for $\alpha_s L \lesssim 1$.

The ABC's of $d\sigma/d^2\mathbf{P}_T\,d\eta$

(Notation of Collins, Soper, Sterman)

$$\frac{d\sigma}{d^2\mathbf{P}_T\,d\eta} =$$

$$\int d^2\mathbf{b}\; e^{i\mathbf{P}_t\cdot\mathbf{b}}$$

$$\times \sum_{a,b} \int dx_A\, f_{a/A}(X_A; 1/b_*) \int dx_A\, f_{b/B}(X_B; 1/b_*)$$

$$\times \exp\left(-\int_{1/b_*^2}^{M_Z^2} \frac{d\mu^2}{\mu^2}\left[A\left(\alpha_s(\mu)\right)\ln\left(\frac{M_Z^2}{\mu^2}\right) + B\left(\alpha_s(\mu)\right)\right]\right)$$

$$\times \tilde{e}_j^2\; C_{ja}\left(x_A; \alpha_s(1/b_*)\right)\; C_{jb}\left(x_B; \alpha_s(1/b_*)\right)$$

$$\times \exp\left(-g_1(b^2)\ln\left(\frac{M_Z^2}{Q_\circ^2}\right) - g_{j/a}(x_A, b^2) - g_{j/a}(x_A, b^2)\right)$$

$$+ Y\left(P_T, \alpha_s(P_T)\right)$$

- Expanding A, B and C perturbatively gives approximation useful for $\alpha_s L \lesssim 1$.
- Beware the contributions from $\mu < 1$ GeV.
- In this formulation, there is an IR cutoff, $1/b_*$.
- Contributions from $\mu < 1/b_*$ are put into *nonperturbative functions* g.
- These functions are numerically important.
- Does this work? Theory by Ladinsky and Yuan. Experiment by D0.

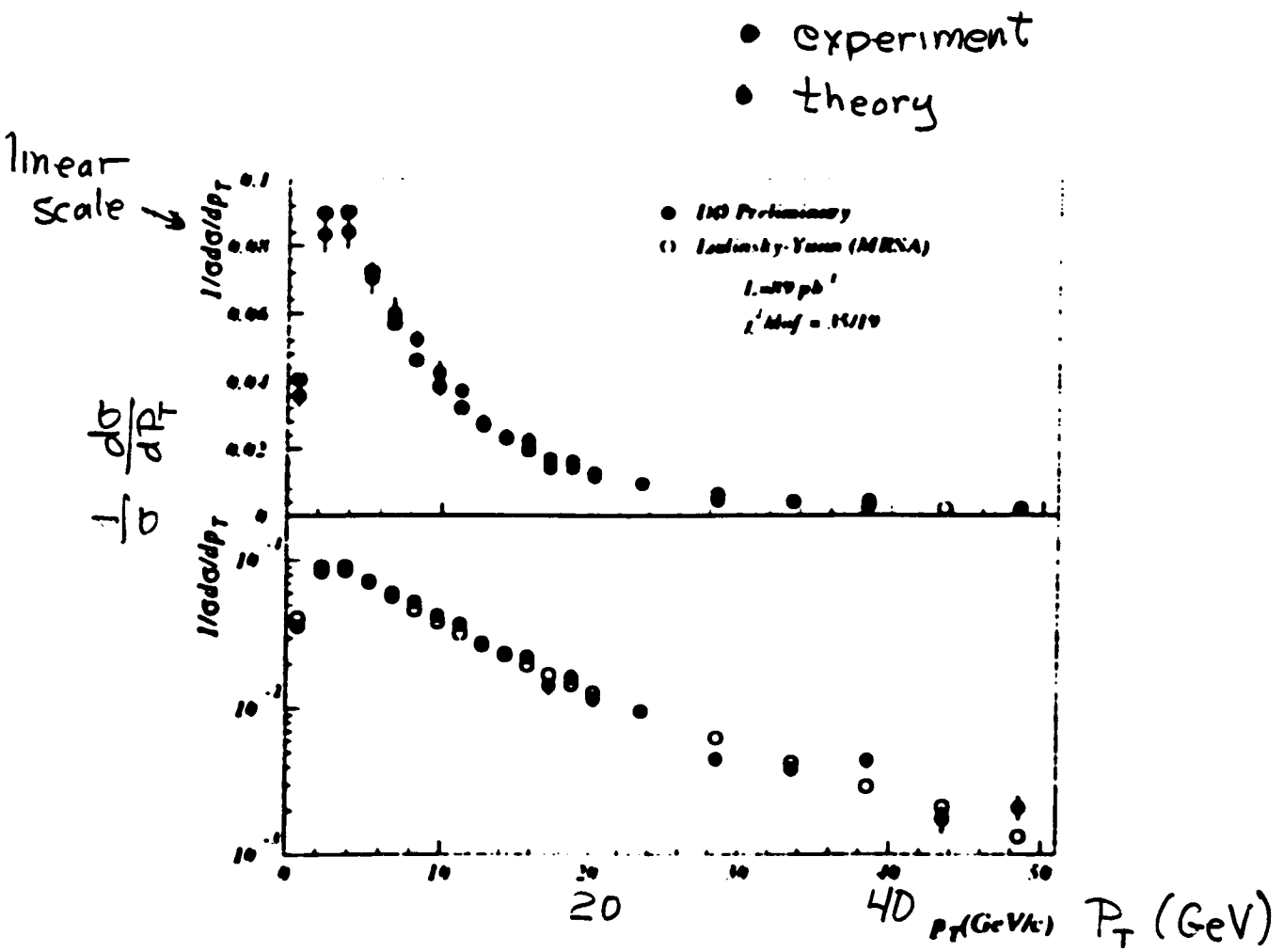


Figure 1: The extracted differential cross section compared to resolution-smeared theory.

Do plot

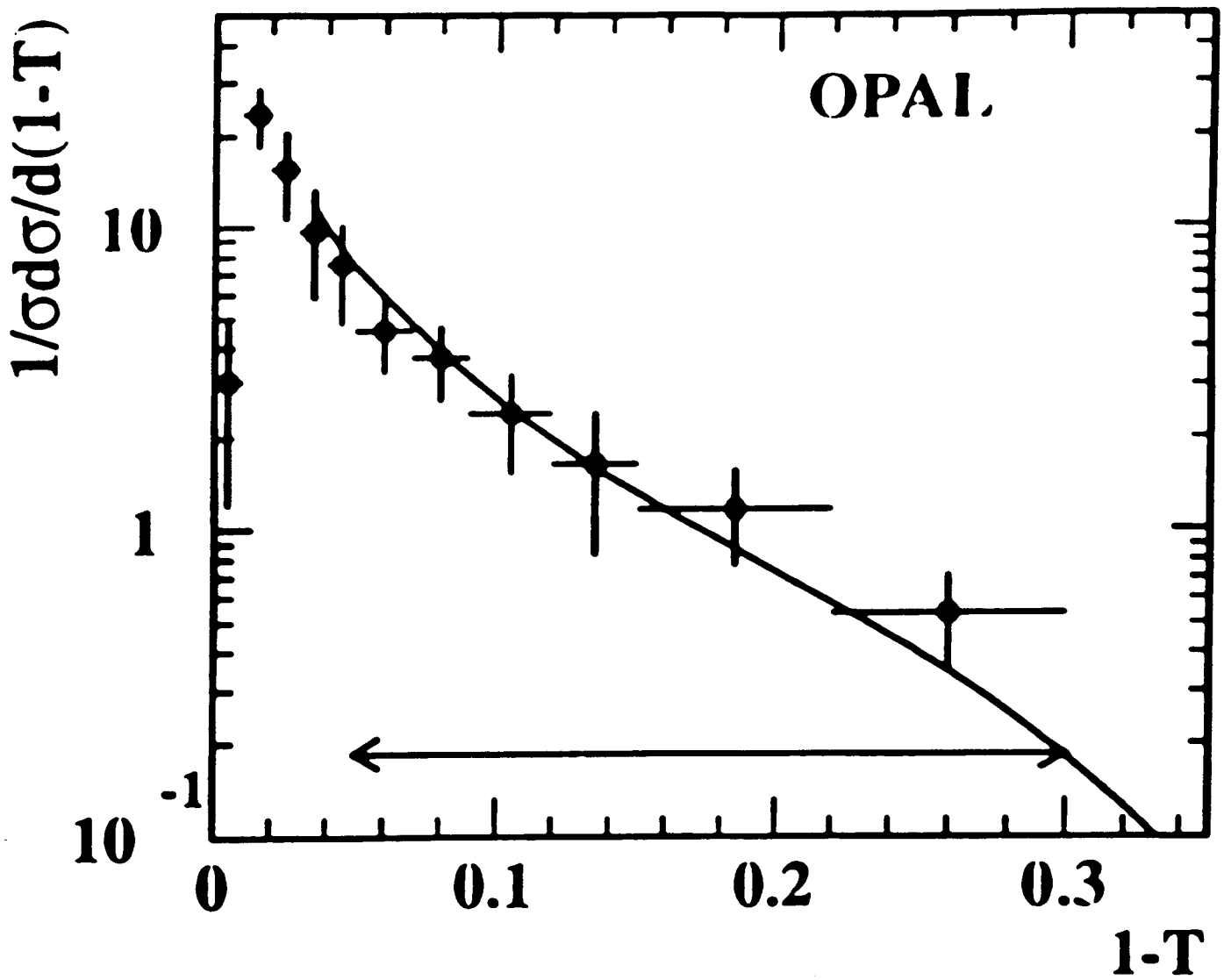
Another double log summation

Thrust distribution in $e^+e^- \to hadrons$, $(1-T) \ll 1$.

$$T = \max_{\hat{n}} \frac{1}{\sqrt{s}} \sum_i |\vec{P}_i \cdot \hat{n}|$$

- $T < 1$
- $(1-T) \ll 1$ for a very two-jet-like event.
- Two logs of $1-T$ for each new α_s.
- Theory: Catani, Trentadue, Turnock & Webber, Nucl. Phys. **B407**, 3 (1993).
- Compare to Opal data ([redacted] nonperturbative contributions).

by Monte Carlo



Opal plot

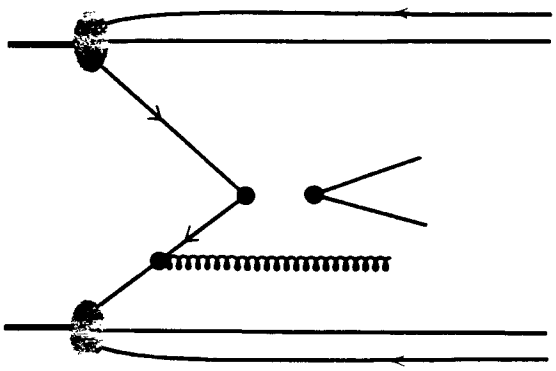
— Resummations

Consider $p + \bar{p} \to A + X$.

- A is what you look at.
  - * $\mu^+\mu^-$ (good theoretical test case);
  - * jets;
  - * heavy quark pairs.
- X is what you don't look at.

Suppose that A need lots of energy:

* minimum x_A required from parton A is large;
* minimum x_B required from parton B is large.



Then radiation of real gluons in the hard process is only barely allowed. Result is large logarithms (two per α_s).

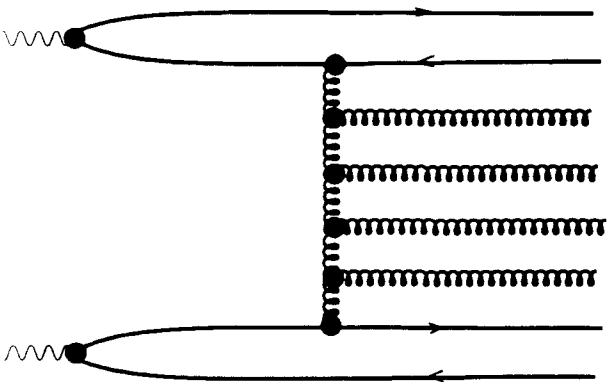
Similar to *thrust* $\to 1$.

We will hear about this at this workshop.

Large rapidity difference summation

E.g. total cross section for scattering two virtual photons

- Virtuality $-q_A \cdot q_A = Q_A^2 \gg 1$ GeV.
- Virtuality $-q_B \cdot q_B = Q_B^2 \gg 1$ GeV.
- $s \gg Q_A^2$, $s \gg Q_B^2$.

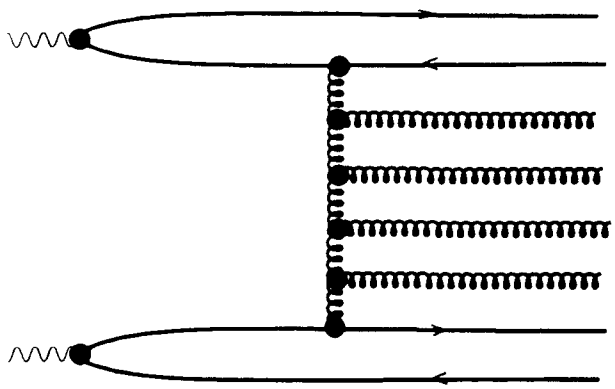


Perturbation theory has logs of $s/Q_{A,B}^2$.

- One log per α_s.
- Logs come from integrating over rapidity y of emitted gluons:

$$\int_{-\ln(\sqrt{s}/Q_B)}^{\ln(\sqrt{s}/Q_A)} dy = \ln\left(\frac{s}{Q_A Q_B}\right)$$

BFKL summation



The virtual γ-γ cross section *à la* BFKL is
(See Hautmann & D.E.S. or de Roeck & Bartels.)

$$\sigma(s,Q_A^2,Q_B^2) = \frac{1}{2\,\pi\,Q_A\,Q_B}\int_{1/2-i\infty}^{1/2+i\infty}\frac{d\,\gamma}{2\pi i}\left(\frac{Q_A^2}{Q_B^2}\right)^{\gamma-1/2} \times\left(\frac{s}{Q_A\,Q_B}\right)^{3\,\chi(\gamma)\,\alpha_s/\pi} V(\gamma)\,V(1-\gamma).$$

where $\chi(\gamma)$ and $V(\gamma)$ are combinations of Γ functions.

- Note the power growth with s.
- This is *leading log* only.
- We need to get beyond leading log.

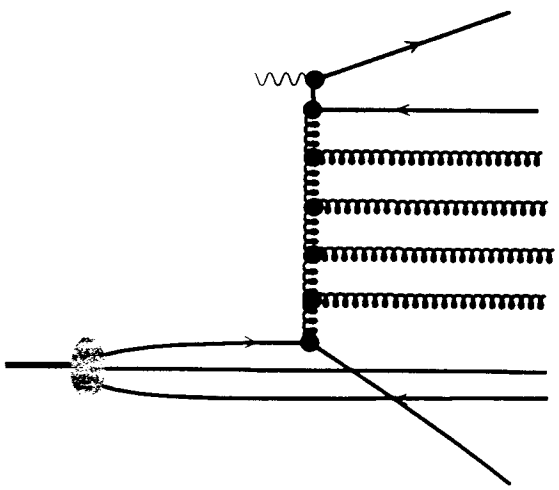
Another BFKL application

Look at DIS with a jet: $e + p \to e + jet + X$

- Virtual photon has large virtuality $-q \cdot q = Q^2$
- Bjorken x is small $x_{bj} \ll 1$.
  $\Rightarrow$ struck parton has small momentum fraction.
- Look for a jet in the final state with $P_T \approx Q$.
- Jet carries a substantial fraction of proton's longitudinal momentum:

  $x_{bj} \ll x_{jet}$

Then there is a large rapidity gap between the DIS subprocess and the jet subprocess. Many gluons can be emitted in this rapidity gap $\Rightarrow$ BFKL dynamics.



Compare

- Order α_s^2 theory [Mirkes & Zeppenfeld];
- BFKL summation
  [Bartels, Del Duca, De Roeck, Graudenz & Wüsthoff];
- Experiment [H1].

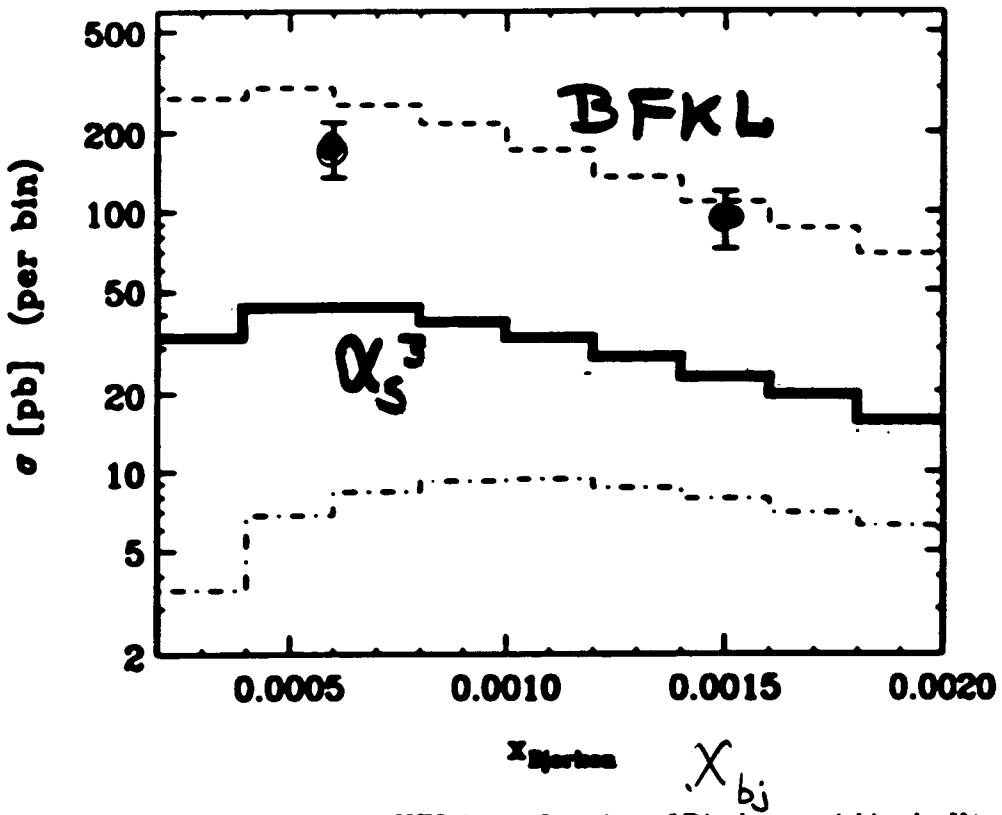


FIG. 4. Forward jet cross section at HERA as a function of Bjorken x within the H1 acceptance cuts [12] (see text). The solid (dash-dotted) histogram gives the NLO (LO) MEPJET result for the scale choice $\mu_R^2 = \mu_F^2 = \xi(0.5\sum k_T)^2$ with $\xi = 1$. The shaded area shows the uncertainty of the NLO prediction, corresponding to a variation of ξ between 0.1 and 10. The BFKL result of Bartels et al. [13] is shown as the dashed histogram. The two data points with error bars correspond to the H1 measurement [12].

Mirkes & Zeppenfeld plot